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Yiren Wang

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Abstract The main purpose of this paper is to study the mean value properties of the Smarandache Superior m -th power part sequence $SSMP(n)$ and the Smarandache Inferior m -th power part sequence $SIMP(n)$, and give several interesting asymptotic formulae for them.

Keywords Smarandache Superior m -th power part sequence, Smarandache Inferior m -th power part sequences, mean value, asymptotic formula.

[illegible]

$$I_n = \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)}.$$

In reference [2], Dr. K.Kashihara asked us to study the properties of these sequences. Gou Su [3] studied these problem, and proved the following conclusion:

For any real number $x > 2$ and integer $m = 2$, we have the asymptotic formula

$$\sum_{n \leq x} SSSP(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right), \quad \sum_{n \leq x} SISP(n) = \frac{x^2}{2} + O\left(x^{\frac{3}{2}}\right),$$

and

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{2}}\right), \quad \lim_{n \rightarrow \infty} \frac{S_n}{I_n} = 1.$$

¹This work is supported by the Shaanxi Provincial Education Department Foundation 08JK433.

In this paper, we shall use the elementary method to give a general conclusion. That is, we shall prove the following:

Theorem 1. Let $m \geq 2$ be an integer, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right),$$

and

$$\sum_{n \leq x} SIMP(n) = \frac{x^2}{2} + O\left(x^{\frac{2m-1}{m}}\right).$$

Theorem 2. For any fixed positive integer $m \geq 2$ and any positive integer n , we have the asymptotic formula

$$S_n - I_n = \frac{m(m-1)}{2m-1} n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

Corollary 1. For any positive integer n , we have the asymptotic formula

$$\frac{S_n}{I_n} = 1 + O\left(n^{-\frac{1}{m}}\right),$$

and the limit $\lim_{n \rightarrow \infty} \frac{S_n}{I_n} = 1$.

Corollary 2. For any positive integer n , we have the asymptotic formula

$$\frac{K_n}{L_n} = 1 + O\left(\frac{1}{n}\right),$$

and the limit $\lim_{n \rightarrow \infty} \frac{K_n}{L_n} = 1$, $\lim_{n \rightarrow \infty} (K_n - L_n) = 0$.

§2. Proof of the theorems

In this section, we shall use the Euler summation formula and the elementary method to complete the proof of our Theorems. For any real number $x > 2$, it is clear that there exists one and only one positive integer M satisfying $M^m < x \leq (M+1)^m$. That is, $M = x^{\frac{1}{m}} + O(1)$. So we have

$$\begin{aligned} \sum_{n \leq x} SSMP(n) &= \sum_{n \leq M^m} SSMP(n) + \sum_{M^m < n \leq x} SSMP(n) \\ &= \sum_{k \leq M} (k^m - (k-1)^m) k^m + ([x] - (M^m + 1))(M+1)^m \\ &= \sum_{k \leq M} (mk^{2m-1} + O(k^{2m-2})) + ([x] - M^m - 1)(M+1)^m \\ &= \frac{m \cdot M^{2m}}{2m} + O(M^{2m-1}) + ([x] - M^m - 1)(M+1)^m \\ &= \frac{M^{2m}}{2} + O(M^{2m-1}). \end{aligned}$$

Note that $M = x^{\frac{1}{m}} + O(1)$, from the above estimate we have the asymptotic formula

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This proves the first formula of Theorem 1.

Now we prove the second one. For any real number $x > 1$, we also have

$$\begin{aligned} \sum_{n \leq x} SIMP(n) &= \sum_{n < M^m} SIMP(n) + \sum_{M^m \leq n \leq x} SIMP(n) \\ &= \sum_{k \leq M} (k^m - (k-1)^m)(k-1)^m + \sum_{M^m \leq n \leq x} M^m \\ &= \sum_{k \leq M} (mk^{2m-1} + O(k^{2m-2})) + ([x] - M^m + 1) M^m \\ &= \frac{M^{2m}}{2} + O(M^{2m-1}) + ([x] - M^m + 1) M^m. \end{aligned}$$

Note that

$$([x] - M^m + 1) M^m \leq M^{2m-1} \leq x^{1-\frac{1}{m}}.$$

Therefore,

$$\sum_{n \leq x} SSMP(n) = \frac{x^2}{2} + O\left(x^{2-\frac{1}{m}}\right).$$

This completes the proof of Theorem 1.

To prove Theorem 2, let $x = n$, then from the method of proving Theorem 1 we have

$$\begin{aligned} S_n - I_n &= \frac{1}{n} (SSMP(1) + SSMP(2) + \cdots + SSMP(n)) \\ &\quad - \frac{1}{n} (SIMP(1) + SIMP(2) + \cdots + SIMP(n)) \\ &= \frac{1}{n} \left(\sum_{k \leq M} (k^m - (k-1)^m) k^m + ([n] - (M^m + 1))(M+1)^m \right) \\ &\quad - \frac{1}{n} \left(\sum_{k \leq M} (k^m - (k-1)^m)(k-1)^m + ([n] - M^m + 1) M^m \right) \\ &= \frac{1}{n} \sum_{k \leq M} m(m-1) k^{2m-2} + O\left(\frac{1}{n} M^{2m-2}\right) \\ &= \frac{m(m-1)}{n(2m-1)} M^{2m-1} + O\left(\frac{1}{n} M^{2m-2}\right). \end{aligned}$$

Note that $M^m < n \leq (M+1)^m$ or $M = n^{\frac{1}{m}} + O(1)$, from the above formula we may immediately deduce that

$$S_n - I_n = \frac{m(m-1)}{2m-1} n^{1-\frac{1}{m}} + O\left(n^{1-\frac{2}{m}}\right).$$

This completes the proof of Theorem 2.

Now we prove the Corollaries. Note that the asymptotic formula

$$I_n = \frac{1}{n}(SIMP(1) + SIMP(2) + \cdots + SIMP(n)) = \frac{1}{n} \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right) \right) = \frac{n}{2} + O\left(n^{1-\frac{1}{m}}\right)$$

and

$$S_n = \frac{1}{n}(SSMP(1) + SSMP(2) + \cdots + SSMP(n)) = \frac{1}{n} \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right) \right) = \frac{n}{2} + O\left(n^{1-\frac{1}{m}}\right).$$

From the above two formula we have

$$\frac{S_n}{I_n} = \frac{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)}{\frac{n}{2} + O\left(n^{\frac{m-1}{m}}\right)} = 1 + O\left(n^{-\frac{1}{m}}\right).$$

Therefore, we have the limit formula

$$\lim_{n \rightarrow \infty} \frac{S_n}{I_n} = 1.$$

Using the same method we can also deduce that

$$K_n = \sqrt[n]{SSMP(1) + SSMP(2) + \cdots + SSMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right) \right)^{\frac{1}{n}}$$

and

$$L_n = \sqrt[n]{SIMP(1) + SIMP(2) + \cdots + SIMP(n)} = \left(\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right) \right)^{\frac{1}{n}}$$

From these formula we may immediately deduce that

$$\frac{K_n}{L_n} = \left(\frac{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)}{\frac{n^2}{2} + O\left(n^{\frac{2m-1}{m}}\right)} \right)^{\frac{1}{n}} = \left(1 + O\left(n^{-\frac{1}{m}}\right) \right)^{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right).$$

Therefore, we have the limit formula

$$\lim_{n \rightarrow \infty} \frac{K_n}{L_n} = 1.$$

Note that $\lim_{n \rightarrow \infty} K_n = \lim_{n \rightarrow \infty} L_n = 1$, we may immediately deduce that

$$\lim_{n \rightarrow \infty} (K_n - L_n) = 0.$$

This completes the proof of Corollary 2.

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A family of Beta-Fibonacci sequences

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Abstract This paper gives a generalization of Beta-nacci and Fibonacci sequences and the general solution obtained is given in terms of Beta-Fibonacci numbers.

Keywords Beta-nacci sequence, Fibonacci sequence.

§1. Preliminaries and introduction

The Fibonacci sequence, say $\{F_n\}_{n=0}^{\infty}$ is defined recurrently by

$$F_n = F_{n-1} + F_{n-2}, \quad \text{for all } n \geq 2, \quad (1)$$

with initial conditions

$$F_0 = 1; \quad F_1 = 1.$$

The general term of the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1}.$$

The Fibonacci sequence has been studied extensively and generalized in many ways. In [1] Peter R. J. Asveld studied the class of recurrence relations

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^k \alpha_j n^j \quad (2)$$

with initial conditions

$$G_0 = 1; \quad G_1 = 1.$$

The main result of [1] consists of an expression for G_n in terms of the Fibonacci numbers F_n and F_{n-1} , and in the parameters $\alpha_0, \alpha_1, \dots, \alpha_k$.

The Beta-nacci sequence, say $\{B_n\}_{n=0}^{\infty}$ is defined recurrently by

$$B_n = B_{n-1} + 2B_{n-2}, \quad \text{for all } n \geq 2, \quad (3)$$

with initial conditions

$$B_0 = 1; \quad B_1 = 1.$$

The general term of the Beta-nacci sequence is

$$B_n = \frac{2^{n+1} + (-1)^n}{3}.$$

In this paper, we give a generalized of Beta-nacci sequences and the general solution obtained is given in terms of Beta-nacci numbers. Also we consider a generalization of the Fibonacci and the Beta-nacci sequences, then we define a new recurrence, which we call the Beta-Fibonacci sequence. Further, we give a generalization of Beta-Fibonacci sequence, called the generalized BF-nacci sequence and express the n^{th} term of the generalized BF-nacci sequence in terms of the Beta-Fibonacci numbers.

§2. A generalization of Beta-nacci sequence

Definition. For any non-negative integer k and any real numbers $\alpha_0, \alpha_1, \dots, \alpha_k$, a generalization of Beta-nacci sequence $\{S_n\}_{n=0}^\infty$ is defined recurrently by

$$S_n = S_{n-1} + 2S_{n-2} + \sum_{j=0}^k \alpha_j n^j, \quad \text{for all } n \geq 2, \quad (4)$$

with initial conditions

$$S_0 = 1; S_1 = 1.$$

Theorem 1. The solution of (4) can be express as

$$S_n = (1 - \Lambda_k)B_n + \Psi_k B_{n-1} + \sum_{j=0}^k p_j(n)\alpha_j, \quad (5)$$

where

- (i) Λ_k is a linear combination of $\alpha_0, \alpha_1, \dots, \alpha_k$;
- (ii) Ψ_k is a linear combination of $\alpha_1, \dots, \alpha_k$;
- (iii) for each j ($0 \leq j \leq k$), $p_j(n)$ is a polynomail of degree j .

Proof of Theorem 1. First, we solve the homogeneous recurrence relation

$$S_n = S_{n-1} + 2S_{n-2}.$$

The characteristic polynomail, $x^2 - x - 2$, has distinct roots 2 and -1 , so the solution is

$$S_n^{(h)} = c_1 2^n + c_2 (-1)^n,$$

where c_1 and c_2 are constants.

Next we find the particular solution of (4).

We set $S_n^{(p)} = \sum_{i=0}^k A_i n^i$, and attempt to determine A_0, A_1, \dots, A_k .

Putting this expression for $S_n^{(p)}$ in (4), we obtain

$$\sum_{i=0}^k A_i n^i = \sum_{i=0}^k A_i (n-1)^i + 2 \sum_{i=0}^k A_i (n-2)^i + \sum_{i=0}^k \alpha_i n^i.$$

Hence, for each $i = 0, 1, \dots, k$, we get

$$A_i - \sum_{m=i}^k \beta_{im} A_m - \alpha_i = 0 \quad (6)$$

with, for $m \geq i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1 + 2^{m-i+1}).$$

From the recurrence relation (6), we can successively determine A_k, A_{k-1}, \dots, A_0 : the coefficient A_i is a linear combination of $\alpha_i, \alpha_{i+1}, \dots, \alpha_k$.

Therefore, we set

$$A_i = \sum_{j=i}^k c_{ij} \alpha_j, \quad (7)$$

which yields, together with (6),

$$\sum_{j=i}^k c_{ij} \alpha_j - \sum_{m=i}^k \beta_{im} \left(\sum_{j=m}^k c_{mj} \alpha_j \right) - \alpha_i = 0.$$

Thus, for $0 \leq i \leq j \leq k$, we have

$$c_{jj} = -\frac{1}{2}$$

$$c_{ij} = -\frac{1}{2} \left(\sum_{m=i+1}^j \beta_{im} c_{mj} \right) \text{ for } i < j.$$

Therefore the particular solution $S_n^{(p)}$ of (4), we obtain

$$\begin{aligned} S_n^{(p)} &= \sum_{i=0}^k \left(\sum_{j=i}^k c_{ij} \alpha_j \right) n^i \\ &= \sum_{j=0}^k \left(\sum_{i=0}^j c_{ij} n^i \right) \alpha_j. \end{aligned}$$

Finally, the recurrence relation (4) has the solution

$$\begin{aligned} S_n &= S_n^{(h)} + S_n^{(p)} \\ &= c_1 2^n + c_2 (-1)^n + \sum_{j=0}^k \left(\sum_{i=0}^j c_{ij} n^i \right) \alpha_j. \end{aligned}$$

The initial conditions: $S_0 = 1; S_1 = 1$, give

$$c_1 = \frac{2}{3} - \frac{1}{3} \left(2\Lambda_k - \Psi_k \right)$$

$$c_2 = \frac{1}{3} - \frac{1}{3} \left(\Lambda_k + \Psi_k \right),$$

where

$$\Lambda_k = \sum_{j=0}^k c_{0j} \alpha_j$$

and

$$\Psi_k = \begin{cases} 0, & \text{if } k = 0; \\ -\sum_{j=1}^k \left(\sum_{i=1}^j c_{ij} \right) \alpha_j, & \text{if } k > 0. \end{cases}$$

Since $B_n = \frac{2^{n+1} + (-1)^n}{3}$, S_n can be written as

$$S_n = (1 - \Lambda_k) B_n + \Psi_k B_{n-1} + \sum_{j=0}^k p_j(n) \alpha_j,$$

where

$$p_j(n) = \sum_{i=0}^j c_{ij} n^i.$$

The proof of the Theorem is now complete.

§3. A generalization of the Fibonacci and Beta-nacci sequences

In this section, we consider a generalization of the Fibonacci and Beta-nacci sequences. First, we define the Beta - Fibonacci sequence as follows:

Definition. Let r be a non-negative integer such that $r \geq 0$. Define the Beta-Fibonacci sequence $\{T_n\}_{n=0}^\infty$ as shown:

$$T_n = T_{n-1} + 2^r T_{n-2} \quad \text{for all } n \geq 2, \quad (8)$$

with initial conditions

$$T_0 = 1; \quad T_1 = 1.$$

When $r = 0$, then the sequence $\{T_n\}_{n=0}^{\infty}$ is reduced to the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ and when $r = 1$, the sequence $\{T_n\}_{n=0}^{\infty}$ is reduced to the Beta-nacci sequence $\{B_n\}_{n=0}^{\infty}$.

The general term of the Beta-Fibonacci sequence is

$$T_n = \frac{1}{\sqrt{1+2^{r+2}}} \left(\phi_1^{n+1} - \phi_2^{n+1} \right),$$

where

$$\phi_1 = \frac{1}{2}(1 + \sqrt{1+2^{r+2}}) \quad \text{and} \quad \phi_2 = \frac{1}{2}(1 - \sqrt{1+2^{r+2}})$$

Now we define a generalization of the Beta-Fibonacci sequence, we call generalized BF-nacci sequence.

Definition. Let r be a non-negative integer such that $r \geq 0$. For any non-negative integer k and any real numbers $\alpha_0, \alpha_1, \dots, \alpha_k$, a generalized BF-nacci sequence $\{R_n\}_{n=0}^{\infty}$ is defined recurrently by

$$R_n = R_{n-1} + 2^r R_{n-2} + \sum_{j=0}^k \alpha_j n^j, \quad \text{for all } n \geq 2, \quad (9)$$

with initial conditions

$$R_0 = 1; \quad R_1 = 1.$$

Note that if we take $r = 0$ in the definition, the sequence $\{R_n\}_{n=0}^{\infty}$ is reduced to the generalization of the Fibonacci sequence $\{G_n\}_{n=0}^{\infty}$ as shown in [1]. Also when $r = 1$, the sequence $\{R_n\}_{n=0}^{\infty}$ is reduced to the generalization of the Beta - nacci sequence $\{S_n\}_{n=0}^{\infty}$.

Theorem 2. The solution of (9) can be express as

$$R_n = (1 - \Lambda_k)R_n + \Psi_k R_{n-1} + \sum_{j=0}^k p_j(n) \alpha_j, \quad (10)$$

where

- (i) Λ_k is a linear combination of $\alpha_0, \alpha_1, \dots, \alpha_k$;
- (ii) Ψ_k is a linear combination of $\alpha_1, \dots, \alpha_k$;
- (iii) for each j ($0 \leq j \leq k$), $p_j(n)$ is a polynomial of degree j .

Proof of Theorem 2. As usual the solution $R_n^{(h)}$ of the homogeneous equation corresponding to (9) is

$$R_n^{(h)} = c_1 \phi_1^n + c_2 \phi_2^n,$$

where

$$\phi_1 = \frac{1}{2}(1 + \sqrt{1+2^{r+2}}) \quad \text{and} \quad \phi_2 = \frac{1}{2}(1 - \sqrt{1+2^{r+2}}).$$

Next, we find the particular solution of (9). We set $R_n^{(p)} = \sum_{i=0}^k A_i n^i$, which yields

$$\sum_{i=0}^k A_i n^i = \sum_{i=0}^k A_i (n-1)^i + 2^r \sum_{i=0}^k A_i (n-2)^i + \sum_{i=0}^k \alpha_i n^i.$$

Thus, for each $i = 0, 1, \dots, k$, we have

$$A_i - \sum_{m=i}^k \beta_{im} A_m - \alpha_i = 0 \quad (11)$$

with, for $m \geq i$,

$$\beta_{im} = \binom{m}{i} (-1)^{m-i} (1 + 2^{m-i+r}).$$

From the recurrence relation (11), we can successively determine A_k, A_{k-1}, \dots, A_0 : the coefficient A_i is a linear combination of $\alpha_i, \alpha_{i+1}, \dots, \alpha_k$.

Therefore, we set

$$A_i = \sum_{j=i}^k c_{ij} \alpha_j, \quad (12)$$

which yields, together with (11),

$$\sum_{j=i}^k c_{ij} \alpha_j - \sum_{m=i}^k \beta_{im} \left(\sum_{j=m}^k c_{mj} \alpha_j \right) - \alpha_i = 0.$$

Thus, for $0 \leq i \leq j \leq k$, we have

$$c_{jj} = -\frac{1}{2^r}$$

$$c_{ij} = -\frac{1}{2^r} \left(\sum_{m=i+1}^j \beta_{im} c_{mj} \right) \quad \text{for } i < j.$$

Hence, for the particular solution $R_n^{(p)}$ of (9), we obtain

$$\begin{aligned} R_n^{(p)} &= \sum_{i=0}^k \left(\sum_{j=i}^k c_{ij} \alpha_j \right) n^i \\ &= \sum_{j=0}^k \left(\sum_{i=0}^j c_{ij} n^i \right) \alpha_j. \end{aligned}$$

Finally, the recurrence relation (9) has the solution

$$\begin{aligned} R_n &= R_n^{(h)} + R_n^{(p)} \\ &= c_1 \phi_1^n + c_2 \phi_2^n + \sum_{j=0}^k \left(\sum_{i=0}^j c_{ij} n^i \right) \alpha_j. \end{aligned}$$

The initial conditions: $R_0 = 1$; $R_1 = 1$, give

$$c_1 = \frac{1}{\sqrt{1+2^{r+2}}} \left((1 - R_0^{(p)}) \phi_1 + R_0^{(p)} - R_1^{(p)} \right)$$

$$c_2 = -\frac{1}{\sqrt{1+2^{r+2}}} \left((1 - R_0^{(p)}) \phi_2 + R_0^{(p)} - R_1^{(p)} \right)$$

Since $T_n = \frac{1}{\sqrt{1+2^{r+2}}} \left(\phi_1^{n+1} - \phi_2^{n+1} \right)$, R_n can be written as

$$R_n = (1 - R_0^{(p)}) T_n + (R_0^{(p)} - R_1^{(p)}) T_{n-1} + \sum_{j=0}^k p_j(n) \alpha_j,$$

where

$$p_j(n) = \sum_{i=0}^j c_{ij} n^i.$$

Hence the proof is complete.

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The product of divisors minimum and maximum functions

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Abstract Let $T(n)$ denote the product of divisors of the positive integer n . We introduce and study some basic properties involving two functions, which are the minimum, resp. the maximum of certain integers connected with the divisors of $T(n)$.

Keywords Arithmetic functions, product of divisors of an integer.

1. Let $T(n) = \prod_{i|n} i$ denote the product of all divisors of n . The product-of-divisors minimum, resp. maximum functions will be defined by

$$\mathcal{T}(n) = \min\{k \geq 1 : n|T(k)\} \quad (1)$$

and

$$\mathcal{T}_*(n) = \max\{k \geq 1 : T(k)|n\}. \quad (2)$$

There are particular cases of the functions F_f^A, G_g^A defined by

$$F_f^A(n) = \min\{k \in A : n|f(k)\}, \quad (3)$$

and its "dual"

$$G_g^A(n) = \max\{k \in A : g(k)|n\}, \quad (4)$$

where $A \subset \mathbb{N}^*$ is a given set, and $f, g : \mathbb{N}^* \rightarrow \mathbb{N}$ are given functions, introduced in [8] and [9]. For $A = \mathbb{N}^*$, $f(k) = g(k) = k!$ one obtains the Smarandache function $S(n)$, and its dual $S_*(n)$, given by

$$S(n) = \min\{k \geq 1 : n|k!\} \quad (5)$$

and

$$S_*(n) = \max\{k \geq 1 : k!|n\}. \quad (6)$$

The function $S_*(n)$ has been studied in [8], [9], [4], [1], [3]. For $A = \mathbb{N}^*$, $f(k) = g(k) = \varphi(k)$, one obtains the Euler minimum, resp. maximum functions

$$E(n) = \min\{k \geq 1 : n|\varphi(k)\} \quad (7)$$

studied in [6], [8], [13], resp., its dual

$$E_*(n) = \max\{k \geq 1 : \varphi(k)|n\}, \quad (8)$$

studied in [13].

For $A = \mathbb{N}^*$, $f(k) = g(k) = S(k)$ one has the Smarandache minimum and maximum functions

$$S_{min}(n) = \min\{k \geq 1 : n|S(k)\}, \quad (9)$$

$$S_{max}(n) = \max\{k \geq 1 : S(k)|n\}, \quad (10)$$

introduced, and studied in [15]. The divisor minimum function

$$D(n) = \min\{k \geq 1 : n|d(k)\} \quad (11)$$

(where $d(k)$ is the number of divisors of k) appears in [14], while the sum-of-divisors minimum and maximum functions

$$\Sigma(n) = \min\{k \geq 1 : n|\sigma(k)\} \quad (12)$$

$$\Sigma_*(n) = \max\{k \geq 1 : \sigma(k)|n\} \quad (13)$$

have been recently studied in [16].

For functions $Q(n), Q_1(n)$ obtained from (3) for $f(k) = k!$ and $A =$ set of perfect squares, resp. $A =$ set of squarefree numbers, see [10].

2. The aim of this note is to study some properties of the functions $\mathcal{T}(n)$ and $\mathcal{T}_*(n)$ given by (1) and (2). We note that properties of $T(n)$ in connection with "multiplicatively perfect numbers" have been introduced in [11]. For other asymptotic properties of $T(n)$, see [7]. For divisibility properties of $T(\sigma(n))$ with $T(n)$, see [5]. For asymptotic results of sums of type $\sum_{n \leq x} \frac{1}{T(n)}$, see [17].

A divisor i of n is called "unitary" if $(i, \frac{n}{i}) = 1$. Let $T^*(n)$ be the product of unitary divisors of n . For similar results to [11] for $T^*(n)$, or $T^{**}(n)$ (i.e. the product of "bi-unitary" divisors of n), see [2]. The product of "exponential" divisors $T_e(n)$ is introduced in paper [12]. Clearly, one can introduce functions of type (1) and (2) for $T(n)$ replaced with one of the above functions $T^*(n), T^{**}, T_e(n)$, but these functions will be studied in another paper.

3. The following auxiliary result will be important in what follows.

Lemma 1.

$$T(n) = n^{d(n)/2}, \quad (14)$$

where $d(n)$ is the number of divisors of n .

Proof. This is well-known, see e.g. [11].

Lemma 2.

$$T(a)|T(b), \text{ if } a|b. \quad (15)$$

Proof. If $a|b$, then for any $d|a$ one has $d|b$, so $T(a)|T(b)$. Reciprocally, if $T(a)|T(b)$, let $\gamma_p(a)$ be the exponent of the prime in a . Clearly, if $p|a$, then $p|b$, otherwise $T(a)|T(b)$ is impossible. If $p^{\gamma_p(b)} \nmid b$, then we must have $\gamma_p(a) \leq \gamma_p(b)$. Writing this fact for all prime divisors of a , we get $a|b$.

Theorem 1. If n is squarefree, then

$$\mathcal{T}(n) = n. \quad (16)$$

Proof. Let $n = p_1 p_2 \dots p_r$, where p_i ($i = \overline{1, r}$) are distinct primes. The relation $p_1 p_2 \dots p_r | T(k)$ gives $p_i | T(k)$, so there is a $d | k$, so that $p_i | d$. But then $p_i | k$ for all $i = \overline{1, r}$, thus $p_1 p_2 \dots p_r = n | k$. Since $p_1 p_2 \dots p_k | T(p_1 p_2 \dots p_k)$, the least k is exactly $p_1 p_2 \dots p_r$, proving (16).

Remark. Thus, if p is a prime, $T(p) = p$; if $p < q$ are primes, then $T(pq) = pq$, etc.

Theorem 2. If $a | b$, $a \neq b$ and b is squarefree, then

$$T(ab) = b. \quad (17)$$

Proof. If $a | b$, $a \neq b$, then clearly $T(b) = \prod_{d|b} d$ is divisible by ab , so $T(ab) \leq b$. Reciprocally, if $ab | T(k)$, let $p | b$ a prime divisor of b . Then $p | T(k)$, so (see the proof of Theorem 1) $p | k$. But b being squarefree (i.e. a product of distinct primes), this implies $b | k$. The least such k is clearly $k = b$.

For example, $T(12) = T(2 \cdot 6) = 6$, $T(18) = T(3 \cdot 6) = 6$, $T(20) = T(2 \cdot 10) = 10$.

Theorem 3. $T(T(n)) = n$ for all $n \geq 1$. (18)

Proof. Let $T(n) | T(k)$. Then by (15) one can write $n | k$. The least k with this property is $k = n$, proving relation (18).

Theorem 4. Let p_i ($i = \overline{1, r}$) be distinct primes, and $\alpha_i \geq 1$ positive integers. Then

$$\begin{aligned} \max \left\{ T \left(\prod_{i=1}^r p_i^{\alpha_i} \right) : i = \overline{1, r} \right\} &\leq T \left(\prod_{i=1}^r p_i^{\alpha_i} \right) \leq \\ &\leq l.c.m. [T(p_1^{\alpha_1}), \dots, T(p_r^{\alpha_r})]. \end{aligned} \quad (19)$$

Proof. In [13] it is proved that for $A = \mathbb{N}^*$, and any function f such that $F_f^{\mathbb{N}^*}(n) = F_f(n)$ is well defined, one has

$$\max \{ F_f(p_i^{\alpha_i}) : i = \overline{1, r} \} \leq F_f \left(\prod_{i=1}^r p_i^{\alpha_i} \right). \quad (20)$$

On the other hand, if f satisfies the property

$$a | b \implies f(a) | f(b) \quad (a, b \geq 1), \quad (21)$$

then

$$F_f \left(\prod_{i=1}^r p_i^{\alpha_i} \right) \leq l.c.m. [F_f(p_1^{\alpha_1}), \dots, F_f(p_r^{\alpha_r})]. \quad (22)$$

By Lemma 2, (21) is true for $f(a) = T(a)$, and by using (20), (22), relation (19) follows.

Theorem 5.

$$T(2^n) = 2^\alpha, \quad (23)$$

where α is the least positive integer such that

$$\frac{\alpha(\alpha+1)}{2} \geq n. \quad (24)$$

Proof. By (14), $2^n | T(k)$ iff $2^n | k^{d(k)/2}$. Let $k = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, when $d(k) = (\alpha_1+1) \dots (\alpha_r+1)$. Since $2^{2n} | k^{d(k)} = p_1^{\alpha_1(\alpha_1+1) \dots (\alpha_r+1)} \dots p_r^{\alpha_r(\alpha_r+1) \dots (\alpha_r+1)}$ (let $p_1 < p_2 < \dots < p_r$), clearly $p_1 = 2$

and the least k is when $\alpha_2 = \dots = \alpha_r = 0$ and α_1 is the least positive integer with $2n \leq \alpha_1(\alpha_1 + 1)$. This proves (23), with (24).

For example, $\mathcal{T}(2^2) = 4$, since $\alpha = 2$, $\mathcal{T}(2^3) = 4$ again, $\mathcal{T}(2^4) = 8$ since $\alpha = 3$, etc.

For odd prime powers, the things are more complicated. For example, for 3^n one has:

Theorem 6.

$$\mathcal{T}(3^n) = \min\{3^{\alpha_1}, 2 \cdot 3^{\alpha_2}\}, \quad (25)$$

where α_1 is the least positive integer such that $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$, and α_2 is the least positive integer such that $\alpha_2(\alpha_2 + 1) \geq n$.

Proof. As in the proof of Theorem 5,

$$3^{2n} | p_1^{\alpha_1(\alpha_1+1)\dots(\alpha_r+1)} \cdot p_2^{\alpha_2(\alpha_2+1)\dots(\alpha_1+1)} \dots p_r^{\alpha_r(\alpha_r+1)\dots(\alpha_1+1)},$$

where $p_1 < p_2 < \dots < p_r$, so we can distinguish two cases:

a) $p_1 = 2$, $p_2 = 3$, $p_3 \geq 5$;

b) $p_1 = 3$, $p_2 \geq 5$.

Then $k = 2^{\alpha_1} \cdot 3^{\alpha_2} \dots p_r^{\alpha_r} \geq 2^{\alpha_1} \cdot 3^{\alpha_2}$ in case a), and $k \geq 3^{\alpha_1}$ in case b). So for the least k we must have $\alpha_2(\alpha_2 + 1)(\alpha_2 + 1) \geq 2n$ with $\alpha_1 = 1$ in case a), and $\alpha_1(\alpha_1 + 1) \geq 2n$ in case b). Therefore $\frac{\alpha_1(\alpha_1+1)}{2} \geq n$ and $\alpha_2(\alpha_2 + 1) \geq n$, and we must select k with the least of 3^{α_1} and $2^1 \cdot 3^{\alpha_2}$, so Theorem 6 follows.

For example, $\mathcal{T}(3^2) = 6$ since for $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, and $\min\{2 \cdot 3^1, 3^2\} = 6$; $\mathcal{T}(3^3) = 9$ since for $n = 3$, $\alpha_1 = 2$, $\alpha_2 = 2$ and $\min\{2 \cdot 3^2, 3^2\} = 9$.

Theorem 7. Let $f : [1, \infty) \rightarrow [0, \infty)$ be given by $f(x) = \sqrt{x} \log x$. Then

$$f^{-1}(\log n) < \mathcal{T}(n) \leq n, \quad (26)$$

for all $n \geq 1$, where f^{-1} denotes the inverse function of f .

Proof. Since $n | \mathcal{T}(n)$, the right side of (26) follows by definition (1) of $\mathcal{T}(n)$. On the other hand, by the known inequality $d(k) < 2\sqrt{k}$, and Lemma 1 (see (14)) we get $\mathcal{T}(k) < k^{\sqrt{k}}$, so $\log \mathcal{T}(k) < \sqrt{k} \log k = f(k)$. Since $n | \mathcal{T}(k)$ implies $n \leq \mathcal{T}(k)$, so $\log n \leq \log \mathcal{T}(k) < f(k)$, and the function f being strictly increasing and continuous, by the bijectivity of f , the left side of (26) follows.

4. The function $\mathcal{T}_*(n)$ given by (2) differs in many aspects from $\mathcal{T}(n)$. The first such property is:

Theorem 8. $\mathcal{T}_*(n) \leq n$ for all n , with equality only if $n = 1$ or $n = \text{prime}$.

Proof. If $\mathcal{T}(k) | n$, then $\mathcal{T}(k) \leq n$. But $\mathcal{T}(k) \geq k$, so $k \leq n$, and the inequality follows.

Let us now suppose that for $n > 1$, $\mathcal{T}_*(n) = n$. Then $\mathcal{T}(n) | n$, by definition 2. On the other hand, clearly $n | \mathcal{T}(n)$, so $\mathcal{T}(n) = n$. This is possible only when $n = \text{prime}$.

Remark. Therefore the equality

$$\mathcal{T}_*(n) = n(n > 1)$$

is a characterization of the prime numbers.

Lemma 3. Let p_1, \dots, p_r be given distinct primes ($r \geq 1$). Then the equation

$$\mathcal{T}(k) = p_1 p_2 \dots p_r$$

is solvable if $r = 1$.

Proof. Since $p_i | T(k)$, we get $p_i | k$ for all $i = \overline{1, r}$. Thus $p_1 \dots p_r | k$, and Lemma 2 implies $T(p_1 \dots p_r) | T(k) = p_1 \dots p_r$. Since $p_1 \dots p_r | T(p_1 \dots p_r)$, we have $T(p_1 \dots p_r) = p_1 \dots p_r$, which by Theorem 8 is possible only if $r = 1$.

Theorem 9. Let $P(n)$ denote the greatest prime factor of $n > 1$. If n is squarefree, then

$$\mathcal{T}_*(n) = P(n). \quad (27)$$

Proof. Let $n = p_1 p_2 \dots p_r$, where $p_1 < p_2 < \dots < p_r$. If $T(k) | (p_1 \dots p_r)$, then clearly $T(k) \in \{1, p_1, \dots, p_r, p_1 p_2, \dots, p_1 p_2 \dots p_r\}$. By Lemma 3 we cannot have

$$T(k) \in \{p_1 p_2, \dots, p_1 p_2 \dots p_r\},$$

so $T(k) \in \{1, p_1, \dots, p_r\}$, when $k \in \{1, p_1, \dots, p_r\}$. The greatest k is $p_r = P(n)$.

Remark. Therefore $\mathcal{T}_*(pq) = q$ for $p < q$. For example, $\mathcal{T}_*(2 \cdot 7) = 7$, $\mathcal{T}_*(3 \cdot 5) = 5$, $\mathcal{T}_*(3 \cdot 7) = 7$, $\mathcal{T}_*(2 \cdot 11) = 11$, etc.

Theorem 10.

$$\mathcal{T}_*(p^n) = p^\alpha (p = \text{prime}), \quad (28)$$

where α is the greatest integer with the property

$$\frac{\alpha(\alpha+1)}{2} \leq n. \quad (29)$$

Proof. If $T(k) | p^n$, then $T(k) = p^m$ for $m \leq n$. Let q be a prime divisor of k . Then $q = T(q) | T(k) = 2^m$ implies $q = p$, so $k = p^\alpha$. But then $T(k) = p^{\alpha(\alpha+1)/2}$ with α the greatest number such that $\alpha(\alpha+1)/2 \leq n$, which finishes the proof of (28).

For example, $\mathcal{T}_*(4) = 2$, since $\frac{\alpha(\alpha+1)}{2} \leq 2$ gives $\alpha_{max} = 1$.

$\mathcal{T}_*(16) = 4$, since $\frac{\alpha(\alpha+1)}{2} \leq 4$ is satisfied with $\alpha_{max} = 2$.

$\mathcal{T}_*(9) = 3$, and $\mathcal{T}_*(27) = 9$ since $\frac{\alpha(\alpha+1)}{2} \leq 3$ with $\alpha_{max} = 2$.

Theorem 11. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^2 q) = \max\{p, q\}. \quad (30)$$

Proof. If $T(k) | p^2 q$, then $T(k) \in \{1, p, q, p^2, pq, p^2 q\}$. The equations $T(k) = p^2$, $T(k) = pq$, $T(k) = p^2 q$ are impossible. For example, for the first equation, this can be proved as follows. By $p | T(k)$ one has $p | k$, so $k = pm$. Then $p(pm)$ are in $T(k)$, so $m = 1$. But then $T(k) = p \neq p^2$. For the last equation, $k = (pq)m$ and $pqm(pm)(qm)(pqm)$ are in $T(k)$, which is impossible.

Theorem 12. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^3 q) = \max\{p^2, q\}. \quad (31)$$

Proof. As above, $T(k) \in \{1, p, q, pq, p^2 q, p^3 q, p^2, p^3\}$ and $T(k) \in \{pq, p^2 q, p^3 q, p^2\}$ are impossible. But $T(k) = p^3$ by Lemma 1 gives $k^{d(k)} = p^6$, so $k = p^m$, when $d(k) = m + 1$. This gives $m(m+1) = 6$, so $m = 2$. Thus $k = p^2$. Since $p < p^2$ the result follows.

Remark. The equation

$$T(k) = p^s \quad (32)$$

can be solved only if $k^{d(k)} = p^{2s}$, so $k = p^m$ and we get $m(m+1) = 2s$. Therefore $k = p^m$, with $m(m+1) = 2s$, if this is solvable. If s is not a **triangular number**, this is impossible.

Theorem 13. Let p, q be distinct primes. Then

$$\mathcal{T}_*(p^s q) = \begin{cases} \max\{p, q\}, & \text{if } s \text{ is not a triangular number,} \\ \max\{p^n, q\}, & \text{if } s = \frac{m(m+1)}{2}. \end{cases}$$

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Generalized random stability of Jensen type mapping

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Abstract In this paper, we consider Jensen type mapping in the setting of generalized random normed spaces. We generalize a Hyers-Ulam stability result in the framework of classical normed spaces.

Keywords Jensen type mapping, generalized random normed space.

§1. Introduction

In 1941 D.H. Hyers [5] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. In 1951 D.G. Bourgin [2] was the second author to treat the Ulam stability problem for additive mappings. In 1978 P.M. Gruber [4] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types.

We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [10] used a stability property of the functional equation

$$f(x - y) + f(x + y) = 2f(x) \quad (0.1)$$

to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials. In 2003–2006 J.M. Rassias and M.J. Rassias [6] and J.M. Rassias [7] solved the above Ulam problem for Jensen and Jensen type mappings. In this paper we consider the stability of Jensen type mapping in the setting of intuitionistic fuzzy normed spaces.

§2. Preliminaries

In the sequel, we shall adopt the usual terminology, notations and conventions of the theory of intuitionistic random normed spaces as in [8].

Definition 1. A measure distribution function is a function $\mu : \mathbb{R} \rightarrow [0, 1]$ which is left continuous on \mathbb{R} , non-decreasing and $\inf_{t \in \mathbb{R}} \mu(t) = 0$, $\sup_{t \in \mathbb{R}} \mu(t) = 1$.

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\mu : X \rightarrow D$ is called a *probabilistic measure* on X and $\mu(x)$ is denoted by μ_x .

Definition 2. A non-measure distribution function is a function $\nu : \mathbb{R} \rightarrow [0, 1]$ which is right continuous on \mathbb{R} , non-increasing and $\inf_{t \in \mathbb{R}} \nu(t) = 1$, $\sup_{t \in \mathbb{R}} \nu(t) = 0$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, then $\nu : X \rightarrow B$ is called a *probabilistic non-measure* on X and $\nu(x)$ is denoted by ν_x .

lemma 1. [1,3] Consider the set L^* and operation \leq_{L^*} defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then (L^*, \leq_{L^*}) is a complete lattice.

We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, a triangular norm $*$ on $[0, 1]$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \rightarrow [0, 1]$ satisfying $T(1, x) = 1 * x = x$ for all $x \in [0, 1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \rightarrow [0, 1]$ satisfying $S(0, x) = 0 \diamond x = x$ for all $x \in [0, 1]$.

Using the lattice (L^*, \leq_{L^*}) , these definitions can be straightforwardly extended.

Definition 3. [3] A triangular norm (t -norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \rightarrow L^*$ satisfying the following conditions:

- (a) $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$ (boundary condition);
- (b) $(\forall (x, y) \in (L^*)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
- (c) $(\forall (x, y, z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
- (d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \implies \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*} , then \mathcal{T} is said to be a continuous t -norm.

Definition 4. [3] A continuous t -norm \mathcal{T} on L^* is said to be continuous t -representable if there exist a continuous t -norm $*$ and a continuous t -conorm \diamond on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ are continuous t -representable.

Now, we define a sequence \mathcal{T}^n recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \dots, x^{(n)}), x^{(n+1)}), \quad \forall n \geq 2, x^{(i)} \in L^*.$$

Definition 5. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \rightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then \mathcal{N} is called an involutive negator. A *negator* on $[0, 1]$ is a decreasing mapping $N : [0, 1] \rightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by

$$N_s(x) = 1 - x, \quad \forall x \in [0, 1].$$

Definition 6. Let μ and ν be measure and non-measure distribution functions from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The triple $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be an intuitionistic random normed space (briefly IRN-space) if X is a vector space, \mathcal{T} is a continuous t -representable and $\mathcal{P}_{\mu, \nu}$ is a mapping $X \times (0, +\infty) \rightarrow L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

- (a) $\mathcal{P}_{\mu, \nu}(x, 0) = 0_{L^*}$;
- (b) $\mathcal{P}_{\mu, \nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;
- (c) $\mathcal{P}_{\mu, \nu}(\alpha x, t) = \mathcal{P}_{\mu, \nu}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
- (d) $\mathcal{P}_{\mu, \nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu, \nu}(x, t), \mathcal{P}_{\mu, \nu}(y, s))$.

In this case, $\mathcal{P}_{\mu, \nu}$ is called an intuitionistic random norm. Here,

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)).$$

Example 1. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be measure and non-measure distribution functions defined by

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right), \quad \forall t \in \mathbb{R}^+.$$

Then $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is an IRN-space.

Definition 7. (1) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu, \nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon), \quad \forall n, m \geq n_0,$$

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{\mathcal{P}_{\mu, \nu}} x$) if $\mathcal{P}_{\mu, \nu}(x_n - x, t) \rightarrow 1_{L^*}$ as $n \rightarrow \infty$ for every $t > 0$.

(3) An IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$.

§3. Stability results

Theorem 1. Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IRN-space, $\varphi : X \times X \longrightarrow Z$ be a function such that for some $0 < \alpha < 2$,

$$\mathcal{P}'_{\mu,\nu}(\varphi(2x, 2x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\alpha\varphi(x, x), t) \quad (x \in X, t > 0) \quad (0.2)$$

and $\lim_{n \rightarrow \infty} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 2^n t) = 1_{L^*}$ for all $x, y \in X$ and $t > 0$. Let $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space. If $f : X \rightarrow Y$ is a mapping such that

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(f(x+y) - f(x-y) - 2f(y), t) \\ & \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y), t) \quad (x, y \in X, t > 0) \end{aligned} \quad (0.3)$$

and $f(0) = 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, y), (2 - \alpha)t). \quad (0.4)$$

Proof. Putting $y = x$ in (0.3) we get

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x), 2t) \quad (x \in X, t > 0). \quad (0.5)$$

Replacing x by $2^n x$ in (0.5), and using (0.2) we obtain

$$\begin{aligned} \mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}, t\right) & \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n x), 2 \times 2^n t) \\ & \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{2 \times 2^n}{\alpha^n}\right). \end{aligned} \quad (0.6)$$

Since $\frac{f(2^n x)}{2^n} - f(x) = \sum_{k=0}^{n-1} \left(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}\right)$, by (0.6) we have

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t \sum_{k=0}^{n-1} \frac{\alpha^k}{2 \times 2^k}\right) \geq_{L^*} \mathbf{M}_{k=0}^{n-1} (\mathcal{P}'_{\mu,\nu}(\varphi(x, x), t)) = \mathcal{P}'_{\mu,\nu}(\varphi(x, x), t),$$

that is

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{2 \times 2^k}}\right). \quad (0.7)$$

By replacing x with $2^m x$ in (0.7) we observe that:

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^m x)}{2^m}, t\right) \geq \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k}{2 \times 2^k}}\right). \quad (0.8)$$

Then $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$. Since $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ is a complete IRN-space this sequence convergent to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (0.8) to obtain

$$\mathcal{P}_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq_{L^*} \mathcal{P}'_{\mu,\nu}\left(\varphi(x, x), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{2 \times 2^k}}\right), \quad (0.9)$$

and so for every $\delta > 0$ we have that

$$\begin{aligned} \mathcal{P}_{\mu,\nu}(A(x) - f(x), t + \delta) &\geq_{L^*} \mathbf{M} \left(\mathcal{P}_{\mu,\nu} \left(A(x) - \frac{f(2^n x)}{2^n}, \delta \right), \mathcal{P}_{\mu,\nu} \left(f(x) - \frac{f(2^n x)}{2^n}, t \right) \right) \\ &\geq_{L^*} \mathbf{M} \left(\mathcal{P}_{\mu,\nu} \left(A(x) - \frac{f(2^n x)}{2^n}, \delta \right), \mathcal{P}'_{\mu,\nu} \left(\varphi(x, x), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{2 \times 2^k}} \right) \right). \end{aligned} \quad (0.10)$$

Taking the limit as $n \rightarrow \infty$ and using (0.10) we get

$$\mathcal{P}_{\mu,\nu}(A(x) - f(x), t + \delta) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x), t(2 - \alpha)). \quad (0.11)$$

Since δ was arbitrary, by taking $\delta \rightarrow 0$ in (0.11) we get

$$\mathcal{P}_{\mu,\nu}(A(x) - f(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(x, x), t(2 - \alpha)).$$

Replacing x, y by $2^n x, 2^n y$ in (0.3) to get

$$\begin{aligned} &\mathcal{P}_{\mu,\nu} \left(\frac{f(2^n(x+y))}{2^n} + \frac{f(2^n(x-y))}{2^n} - \frac{2f(2^n y)}{2^n}, t \right) \\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 2^n t), \end{aligned} \quad (0.12)$$

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n \rightarrow \infty} \mathcal{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), 2^n t) = 1_{L^*}$ we conclude that A fulfills (0.1). To Prove the uniqueness of the additive function A , assume that there exists an additive function $A' : X \rightarrow Y$ which satisfies (0.4). Fix $x \in X$. Clearly $A(2^n x) = 2^n A(x)$ and $A'(2^n x) = 2^n A(x)$ for all $n \in \mathbb{N}$. It follows from (0.4) that

$$\begin{aligned} \mathcal{P}_{\mu,\nu}(A(x) - A'(x), t) &= \mathcal{P}_{\mu,\nu} \left(\frac{A(2^n x)}{2^n} - \frac{A'(2^n x)}{2^n}, t \right) \\ &\geq_{L^*} \mathbf{M} \left\{ \mathcal{P}_{\mu,\nu} \left(\frac{A(2^n x)}{2^n} - \frac{f(2^n x)}{2^n}, \frac{t}{2} \right), \mathcal{P}_{\mu,\nu} \left(\frac{A'(2^n x)}{2^n} - \frac{f(2^n x)}{2^n}, \frac{t}{2} \right) \right\} \\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu} \left(\varphi(2^n x, 2^n x), 2^n (2 - \alpha) \frac{t}{2} \right) \\ &\geq_{L^*} \mathcal{P}'_{\mu,\nu} \left(\varphi(x, x), \frac{2^n (2 - \alpha) \frac{t}{2}}{\alpha^n} \right). \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{27^n (27 - \alpha) t}{2 \alpha^n} = \infty$, we get $\lim_{n \rightarrow \infty} \mathcal{P}'_{\mu,\nu}(\varphi(x, 0), \frac{27^n (27 - \alpha) t}{2 \alpha^n}) = 1_{L^*}$. Therefore $\mathcal{P}_{\mu,\nu}(A(x) - A'(x), t) = 1$ for all $t > 0$, whence $A(x) = A'(x)$.

Corollary 1. Let X be a linear space, $(Z, \mathcal{P}'_{\mu,\nu}, \mathbf{M})$ be an IRN-space, $(Y, \mathcal{P}_{\mu,\nu}, \mathbf{M})$ be a complete IRN-space, p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that

$$\mathcal{P}_{\mu,\nu}(f(x+y) + f(x-y) - 2f(y), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}((\|x\|^p + \|y\|^q)z_0, t), \quad (0.13)$$

$x, y \in X, t > 0, f(0) = 0$ and $p, q < 1$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu,\nu}(\|x\|^p z_0, (2 - 2^p)t). \quad (0.14)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 2^p$.

Corollary 2. Let X be a linear space, $(Z, \mathcal{P}'_{\mu, \nu}, \mathbf{M})$ be an IRN-space, $(Y, \mathcal{P}_{\mu, \nu}, \mathbf{M})$ be a complete IRN-space and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that

$$\mathcal{P}_{\mu, \nu}(f(x+y) + f(x-y) - 2f(y), t) \geq_{L^*} \mathcal{P}'_{\mu, \nu}(\varepsilon z_0, t) \quad (0.15)$$

$x, y \in X$, $t > 0$, $f(0) = 0$, then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\mathcal{P}_{\mu, \nu}(f(x) - A(x), t) \geq_{L^*} \mathcal{P}'_{\mu, \nu}(\varepsilon z_0, t). \quad (0.16)$$

for all $x \in X$ and $t > 0$.

Proof. Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then the corollary is followed from Theorem 3.1 by $\alpha = 1$.

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On Isomorphisms of KU-algebras

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Abstract In this paper, we study homomorphisms of KU-algebras and investigate its properties. Moreover, some consequences of the relations between quotient KU-algebras and isomorphisms are shown.

Keywords Homomorphism, isomorphism, ideal, congruence, KU-algebras.

§1. Introduction

In [3], C. Prabpayak and U. Leerawat studied ideals and congruences of BCC-algebras ([1],[2]) and introduced a new algebraic structure which is called KU-algebras. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. The purpose of this paper is to derive some straightforward consequences of the relations between quotient KU-algebras and isomorphisms and also investigate some of its properties.

§2. Preliminaries

A nonempty set G with a constant 0 and a binary operation denoted by juxtaposition is called a KU-algebra if for all $x, y, z \in G$ the following conditions hold:

- (1) $(xy)((yz)(xz)) = 0$,
- (2) $0x = x$,
- (3) $x0 = 0$,
- (4) $xy = 0 = yx$ implies $x = y$,

for all $x, y, z \in G$.

By (1), we get $(00)((0x)(0x)) = 0$. It follows that $xx = 0$ for all $x \in G$. And if we put $y = 0$ in (1), then we obtain $z(xz) = 0$ for all $x, z \in G$.

A subset S of a KU-algebra G is called subalgebra of G if $xy \in S$ whenever $x, y \in S$. A non-empty subset A of a KU-algebra G is called an ideal of G if it satisfies the following conditions:

- (5) $0 \in A$,
- (6) for all $x, y, z \in G$, $x(yz) \in A$ and $y \in A$ imply $xz \in A$.

Putting $x = 0$ in (6), we obtain the following: for all $x, y \in G$, $xy \in A$ and $x \in A$ imply $y \in A$.

On KU-algebra $(G, \cdot, 0)$. We define a binary relation \leq on G by putting $x \leq y$ if and only if $yx = 0$. Then $(G; \leq)$ is a partially ordered set and 0 is its smallest element. Thus a KU-algebra G satisfies conditions: $(yz)(xz) \leq xy$, $0 \leq x$, $x \leq y \leq x$ implies $x = y$.

Let A be an ideal of KU-algebra G . Define the relation \sim on G by $x \sim y$ if and only if $xy \in A$ and $yx \in A$. Then the relation \sim is a congruence on G . And $C_0 = \{x \in G \mid x \sim 0\}$ is an ideal of G .

Let \sim be a congruence relation on a KU-algebra G and let A be an ideal of G . Define A_x by $A_x = \{y \in G \mid y \sim x\} = \{y \in G \mid xy \in A, yx \in A\}$. Then the family $\{A_x : x \in G\}$ gives a partition of G which is denoted by G/A . For any $x, y \in G$, we define $A_x * A_y = A_{xy}$. Since \sim has the substitution property, the operation $*$ is well-defined. It is easily checked that $(G/A, *, A)$ is a KU-algebra.

If A is an ideal of KU-algebra G , then it is clear that $A_x = A_0 = A$ for all x in A .

Let $(G, \cdot, 0)$ and $(H, *, 0)$ be KU-algebras. A homomorphism is a map $f : G \rightarrow H$ satisfying $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G$. An injective homomorphism is called monomorphism, a surjective homomorphism is called epimorphism and a bijective homomorphism is called isomorphism. The kernel of the homomorphism f , denoted by $\ker f$, is the set of elements of G that map to 0 in H .

§3. Results

For an ideal A of a KU-algebra G . Then the canonical mapping $f : G \rightarrow G/A$ defined by $f(x) = A_x$ is an epimorphism and $\ker f$ is an ideal of G (see in [3]).

Definition 1. Let ϕ be a mapping of a KU-algebra G into a KU-algebra H , and let $A \subseteq G$ and $B \subseteq H$. The image of A in H under ϕ is

$$\phi(A) = \{\phi(a) \mid a \in A\}$$

and the inverse image of B in G is

$$\phi^{-1}(B) = \{g \in G \mid \phi(g) \in B\}.$$

Theorem 1. Let ϕ be a homomorphism of a KU-algebras G into a KU-algebra H .

- (i) If 0 is the identity in G , then $\phi(0)$ is the identity in H .
- (ii) If S is a KU-subalgebra of G , then $\phi(S)$ is a KU-subalgebra of H .
- (iii) If A is an ideal of G , then $\phi(A)$ is an ideal in $\phi(G)$.
- (iv) If K is a KU-subalgebra of H , then $\phi^{-1}(K)$ is a KU-subalgebra of G .
- (v) If B is an ideal in $\phi(G)$, then $\phi^{-1}(B)$ is an ideal in G .
- (vi) If ϕ is a homomorphism from a KU-algebra G to a KU-algebra H , then ϕ is 1-1 if and only if $\ker \phi = \{0\}$.

Proof of Theorem 1.

(i) Let 0 be the identity in G and $\acute{0}$ the identity in H . Then $\phi(0)\acute{0} = \acute{0}$ and

$$\begin{aligned}\acute{0}\phi(0) &= (\phi(0)\phi(0))\phi(0) \\ &= \phi(00)\phi(0) \\ &= \phi(0)\phi(0) \\ &= \acute{0}.\end{aligned}$$

By (4), we get that $\phi(0) = \acute{0}$.

(ii) Let S be a KU-subalgebra of G . Let $x, y \in \phi(A)$. That means $x = \phi(a)$ and $y = \phi(b)$ for some a, b in G . Then $xy = \phi(a)\phi(b) = \phi(ab)$. Thus $\phi(A)$ is a KU-subalgebra of H .

(iii) Let A be an ideal of G . Clearly, $\acute{0} \in \phi(A)$. If $\phi(x)(\phi(y)\phi(z)) \in \phi(A)$ and $\phi(y) \in \phi(A)$, then $\phi(x(yz)) \in \phi(A)$ and $\phi(y) \in \phi(A)$, so $x(yz), y \in A$. Since A is an ideal, $xz \in A$. Thus $\phi(x)\phi(z) = \phi(xz) \in \phi(A)$. Hence $\phi(A)$ is an ideal of $\phi(G)$.

(iv) Let K be a KU-subalgebra of H . Let $x, y \in \phi^{-1}(K)$. Then $\phi(x) = a$ and $\phi(y) = b$ for some $a, b \in K$. Thus $\phi(xy) = \phi(x)\phi(y) = ab \in K$, since K is a KU-subalgebra. Hence $xy \in \phi^{-1}(K)$.

(v) Let B be an ideal in $\phi(G)$. Since $\phi(0) = \acute{0}$, $0 \in \phi^{-1}(B)$. Let $x(yz) \in \phi^{-1}(B)$ and $y \in \phi^{-1}(B)$ for $x, y, z \in G$. Then $\phi(x)(\phi(y)\phi(z)) \in B$ and $\phi(y) \in B$. Since B is an ideal of $\phi(G)$, $\phi(x)\phi(z) \in B$. Thus $\phi(xz) \in B$. We get that $xz \in \phi^{-1}B$. Hence $\phi^{-1}B$ is an ideal of G .

(vi) Suppose ϕ is 1-1. Let $x \in \ker\phi$. Then $\phi(x) = \acute{0}$. Since $\phi(0) = \acute{0}$, $\phi(x) = \phi(0)$. Since ϕ is 1-1, $x = 0$. Thus $\ker\phi = \{0\}$.

Conversely, suppose $\ker\phi = \{0\}$. Let $x, y \in G$ be such that $\phi(x) = \phi(y)$. Then we get that

$$\phi(xy) = \phi(x)\phi(y) = \acute{0}$$

and

$$\phi(yx) = \phi(y)\phi(x) = \acute{0}$$

Thus $xy, yx \in \ker\phi$, so $xy = 0 = yx$. It follows that $x = y$. Hence ϕ is 1-1.

Next, we state the first isomorphism of KU-algebras as the following theorem:

Theorem 2. (First Isomorphism Theorem)

If ϕ is an epimorphism from a KU-algebra G onto a KU-algebra H , then the quotient KU-algebra $G/\ker(\phi)$ is isomorphic to H .

Proof of Theorem 2. Let $\phi : G \rightarrow H$ be an epimorphism and $\psi : G/A \rightarrow H$ defined by $\psi(A_x) = \phi(x)$ for all $A_x \in G/A$, where $A = \ker(\phi)$.

Let $A_x, A_y \in G/A$ be such that $A_x = A_y$. Then $A_{xy} = A_x * A_y = A$ and $A_{yx} = A_y * A_x = A$. So $xy, yx \in A$. Thus $\phi(x)\phi(y) = 0 = \phi(y)\phi(x)$. By (4), we get that $\phi(x) = \phi(y)$. It follows that $\psi(A_x) = \psi(A_y)$. Hence ψ is well-defined.

Let $A_x, A_y \in G/A$ be such that $\psi(A_x) = \psi(A_y)$. Then $\phi(x) = \phi(y)$, so $\phi(x)\phi(y) = \acute{0} = \phi(y)\phi(x)$. Since ϕ is a homomorphism, $\phi(xy) = \acute{0} = \phi(yx)$. Thus $xy, yx \in A$, and we have $x \sim y$. Then we get $A_x = A_y$. Hence ψ is an injection. It is obvious that ψ is a surjection.

Since for all $A_x, A_y \in G/A$

$$\begin{aligned}
 \psi(A_x * A_y) &= \psi(A_{xy}) \\
 &= \phi(xy) \\
 &= \phi(x)\phi(y) \\
 &= \psi(A_x)\psi(A_y),
 \end{aligned}$$

ψ is a homomorphism. This completes the proof that ψ is an isomorphism. The map ψ is

$$\begin{array}{ccc}
 G & \xrightarrow{\phi} & H \\
 \gamma \downarrow & \nearrow \psi & \\
 G/\ker\phi & &
 \end{array}$$

a canonical map in the sense that if γ is the canonical homomorphism $\gamma : G \rightarrow G/\ker(\phi)$ of Theorem 2, then

$$\phi = \psi \circ \gamma.$$

Theorem 3. Let X, Y, Z be KU-algebras. Suppose that $\phi : X \rightarrow Y$ is an epimorphism, and let $\psi : X \rightarrow Z$ be a homomorphism. If $\ker(\phi) \subseteq \ker(\psi)$, then there exists a unique homomorphism $\gamma : Y \rightarrow Z$ such that $\gamma \circ \phi = \psi$.

$$\begin{array}{ccc}
 X & \xrightarrow{\phi} & Y \\
 \psi \downarrow & \nearrow \gamma & \\
 Z & &
 \end{array}$$

Proof of Theorem 3. Let $y \in Y$. Since ϕ is onto, there exists $x_y \in X$ such that $\phi(x_y) = y$. Then we define

$$\gamma(y) = \psi(x_y)$$

To show that γ is well defined, let $a, b \in Y$ be such that $a = b$. Since ϕ is onto, $a = \phi(x_a)$ and $b = \phi(x_b)$ for some $x_a, x_b \in X$. Then $\phi(x_a) = \phi(x_b)$, so

$$\phi(x_a x_b) = \phi(x_a)\phi(x_b) = 0$$

and

$$\phi(x_b x_a) = \phi(x_b)\phi(x_a) = 0.$$

Then $x_a x_b, x_b x_a \in \ker(\phi)$. Since $\ker(\phi) \subseteq \ker(\psi)$, $\psi(x_a x_b) = 0$ and $\psi(x_b x_a) = 0$. Then

$$\psi(x_a x_b) = 0 = \psi(x_b x_a).$$

By (4), we get that $\psi(x_a) = \psi(x_b)$. That is $\gamma(a) = \gamma(b)$. Hence γ is well defined.

To show that $\gamma \circ \phi = \psi$, let $x \in X$. Then $\phi(x) = y$ for some $y \in Y$. Now we have $\gamma(y) = \psi(x)$. Thus

$$\begin{aligned} (\gamma \circ \phi)(x) &= \gamma(\phi(x)) \\ &= \gamma(y) \\ &= \psi(x). \end{aligned}$$

Hence $\gamma \circ \phi = \psi$.

Next, we show that γ is homomorphism. Let $a, b \in Y$. Then there exist $x_a, x_b \in X$ such that $a = \phi(x_a)$ and $b = \phi(x_b)$. Thus $\gamma(a) = \psi(x_a)$ and $\gamma(b) = \psi(x_b)$. Now we have

$$ab = \phi(x_a)\phi(x_b) = \phi(x_ax_b).$$

The equation

$$\begin{aligned} \gamma(ab) &= \psi(x_ax_b) \\ &= \psi(x_a)\psi(x_b) \\ &= \gamma(a)\gamma(b) \end{aligned}$$

shows that γ is a homomorphism.

Finally, if $\beta : Y \rightarrow Z$ is another function such that $\beta \circ \phi = \psi$. Then for all $y \in Y$ there exists $x_y \in X$ such that $y = \phi(x_y)$. Thus

$$\begin{aligned} \gamma(y) &= \psi(x_y) \\ &= (\beta \circ \phi)(x_y) \\ &= \beta(\phi(x_y)) \\ &= \beta(y). \end{aligned}$$

This completes the proof.

Corollary 1. Let X, Y , be KU-algebras, let A be an ideal of X , and let ϕ be a canonical mapping from X onto X/A . If ψ is a homomorphism from X to Y and $A \subseteq \ker(\psi)$, then there exists a unique homomorphism $\gamma : X/A \rightarrow Y$ such that $\gamma \circ \phi = \psi$.

Theorem 4. (Second Isomorphism Theorem)

Let G be a KU-algebra. Let A, B be ideals of G . If $A \cup B$ is a KU-algebra, then the quotient KU-algebras $(A \cup B)/B$ and $A/(A \cap B)$ are isomorphic.

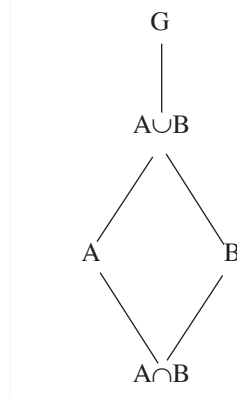
Proof of Theorem 4. Let $\phi : A \rightarrow (A \cup B)/B$ defined by $\phi(x) = B_x$ for all $x \in A$. It is obvious that ϕ is well defined. Let $B_x \in (A \cup B)/B$. If $x \in A$, then $B_x = \phi(x)$. If $x \in B$, then $B_x = B_0 = \phi(0)$. Thus ϕ is onto $(A \cup B)/B$. The equation

$$\begin{aligned} \phi(xy) &= B_{xy} \\ &= B_x * B_y \\ &= \phi(x)\phi(y) \end{aligned}$$

shows that ϕ is a homomorphism.

Now let $x \in \ker\phi$. Then we get $\phi(x) = B_0$, so $B_x = B_0$. Thus $x \in B$. Since $\ker\phi \subseteq A$, $x \in A \cap B$. Hence $\ker\phi \subseteq A \cap B$. On the other hand, let $x \in A \cap B$. Then $x \in B$. Thus $\phi(x) = B_x = B_0$, so $x \in \ker\phi$. Hence $A \cap B \subseteq \ker\phi$. Therefore, $\ker\phi = A \cap B$. Theorem 2. immediately gives us that $(A \cup B)/B \cong A/(A \cap B)$.

We can describe Theorem 4 by the following figure:



Next, we state the third isomorphism theorem of KU-algebras.

Theorem 5. (Third Isomorphism Theorem)

Let G be a KU-algebra. Let A and B be ideals of G , with $A \subseteq B \subseteq G$. Then

- (i) the quotient B/A is an ideal of the quotient G/A , and
- (ii) the quotient KU-algebra $(G/A)/(B/A)$ is isomorphic to G/B .

Proof of Theorem 5.

(i) To show that B/A is an ideal of G/A , let $A_x * (A_y * A_z) \in B/A$ and $A_y \in B/A$. Then $x(yz) \in B$ and $y \in B$. Since B is an ideal of G , $xy \in B$, so $A_{xz} \in B/A$. Thus $A_x * A_z \in B/A$. It is clear that $A \in B/A$. Therefore, B/A is an ideal of G/A .

(ii) Let $\phi : G/A \rightarrow G/B$ defined by $\phi(A_x) = B_x$. Let $A_x = A_y$. Then $x \sim y$ determined by A , that is $xy, yx \in A$. Since $A \subseteq B$, $xy, yx \in B$. Thus $x \sim y$ determined by B , and hence $B_x = B_y$. Then $\phi(A_x) = \phi(A_y)$. Therefore, ϕ is well defined. To show that ϕ is onto G/B , let $B_x \in G/B$. If $x \in G$ and $x \notin B$, then $B_x = \phi(A_x)$. If $x \in B$, then $B_x = B_0 = \phi(B_0)$. Hence ϕ is onto. The equation

$$\begin{aligned}
 \phi(A_x * A_y) &= \phi(A_{xy}) \\
 &= B_{xy} \\
 &= B_x * B_y \\
 &= \phi(A_x) * \phi(A_y)
 \end{aligned}$$

shows that ϕ is a homomorphism.

To show that $\ker\phi = B/A$, let $A_x \in \ker\phi$. Then $\phi(A_x) = B_0$, so $B_x = B_0$. Thus $x \in B$. Now we have $A_x \in B/A$. Hence $\ker\phi \subseteq B/A$. Going the other hand, let $A_x \in B/A$. $\phi(A_x) = B_x = B_0$, since $x \in B$. Thus $A_x \in \ker\phi$, and hence $B/A \subseteq \ker\phi$. Therefore, $\ker\phi = B/A$. By Theorem 2, $(G/A)/(B/A)$ is isomorphic to G/B .

$$\begin{array}{ccc}
 \text{G} & \text{---} & \text{G/A} \\
 | & & | \\
 \text{B} & \text{---} & \text{B/A} \\
 | & & | \\
 \text{A} & \text{---} & \text{A/A}
 \end{array}$$

It turns out that an analogous result of the third isomorphism theorem for groups is also true for KU-algebras.

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Smarandache friendly numbers-another approach

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Abstract One approach to Smarandache friendly numbers is given by A. Murthy, who defined them Ref [1]. Another approach is presented here.

Keywords Smarandache friendly numbers.

Smarandache friendly numbers were defined by A. Murthy [1] as follows.

Definition 1. If the sum of any set of consecutive terms of a sequence is equal to the product of first and last number, then the first and the last numbers are called a pair of Smarandache friendly numbers.

Here, we will consider a sequence of natural numbers.

1. It is easy to note that $(3, 6)$ is a friendly pair as $3 + 4 + 5 + 6 = 18 = 3 \cdot 6$.
2. By elementary operations and trial, we can find such pairs, but as magnitude of natural numbers increases, this work becomes tedious. Hence an algorithm is presented here.

Assume that (m, n) is a pair of friendly numbers, $n > m$, so that

$$m + (m + 1) + (m + 2) + \cdots + n = m \cdot n.$$

Let $n = m + k$, where k is a natural number. Then the above equation becomes

$$m + (m + 1) + (m + 2) + \cdots + n = m \cdot (m + k).$$

On simplification, this gives,

$$k^2 + k - 2(m^2 - m) = 0.$$

That is,

$$k = \frac{-1 + \sqrt{1 + 8(m^2 - m)}}{2},$$

considering the positive sign only.

Now, k will be a natural number only if $1 + 8(m^2 - m)$ is a perfect square of an odd natural number.

For $m = 3$, we have $k = 3$, so that $n = 3 + 3 = 6$, and then $(3, 6)$ are friendly numbers as we observed earlier.

For $m = 5$, $k = \frac{-1+\sqrt{161}}{2}$, which is not an integer. Hence k does not exist for every m .

For $m = 15$, $k = 20$, hence $n = 35$. So the next pair of friendly numbers is $(15, 35)$. Other pairs are $(85, 204)$ and $(493, 1189)$.

At the end, the list of m and $1 + 8(m^2 - m)$ is given using a computer software.

2. If (m, n) is a friendly pair, of natural numbers, then it is conjectured that $(m + 2n, 2m + 5n - 1)$ is also a friendly pair.

Since, $(3, 6)$ and $(15, 35)$ are friendly pairs, we have $3x + 6y = 15$ and $3p + 6q = 35$, for some x, y, p, q being natural numbers. These equations are true for $x = 1, y = 2, p = 2$ and $q = 5$. With 1 to be subtracted from second equation.

These solutions are unique, hence this conjecture.

This suggests that there are infinite pairs of friendly natural numbers.

Definition 2. [1] x and y are primes. They are called Smarandache friendly primes if the sum of any set of consecutive primes, whose first term is x and last is y , is equal to their product $x \cdot y$.

Example. $(2, 5), (3, 13), (5, 31)$ are pairs of friendly primes, for $3 + 5 + 7 + 11 + 13 = 39 = 3 \cdot 13$.

Since the primes are not uniformly distributed, an algorithm for friendly primes seems to be impossible.

4. For various sequences, different pairs of friendly numbers can be obtained.

The values of $m \leq 1000$ and $1 + 8(m^2 - m)$ are given below. Those with perfect squares are underlined. Only four pairs were obtained.

| | | | | | | | | | |
|----|----------|----|----------|----------|---------------|----|----------|-----------|----------------|
| 1 | 1.0000 | 2 | 4.1231 | <u>3</u> | <u>7.0000</u> | 4 | 9.8489 | 5 | 12.6886 |
| 6 | 15.5242 | 7 | 18.3576 | 8 | 21.1896 | 9 | 24.0208 | 10 | 26.8514 |
| 11 | 29.6816 | 12 | 32.5115 | 13 | 35.3412 | 14 | 38.1707 | <u>15</u> | <u>41.0000</u> |
| 16 | 43.8292 | 17 | 46.6583 | 18 | 49.4874 | 19 | 52.3163 | 20 | 55.1453 |
| 21 | 57.9741 | 22 | 60.8030 | 23 | 63.6318 | 24 | 66.4605 | 25 | 69.2892 |
| 26 | 72.1180 | 27 | 74.9466 | 28 | 77.7753 | 29 | 80.6040 | 30 | 83.4326 |
| 31 | 86.2612 | 32 | 89.0898 | 33 | 91.9184 | 34 | 94.7470 | 35 | 97.5756 |
| 36 | 100.4042 | 37 | 103.2327 | 38 | 106.0613 | 39 | 108.8899 | 40 | 111.7184 |
| 41 | 114.5469 | 42 | 117.3755 | 43 | 120.2040 | 44 | 123.0325 | 45 | 125.8610 |
| 46 | 128.6895 | 47 | 131.5181 | 48 | 134.3466 | 49 | 137.1751 | 50 | 140.0036 |
| 51 | 142.8321 | 52 | 145.6606 | 53 | 148.4891 | 54 | 151.3176 | 55 | 154.1460 |
| 56 | 156.9745 | 57 | 159.8030 | 58 | 162.6315 | 59 | 165.4600 | 60 | 168.2884 |
| 61 | 171.1169 | 62 | 173.9454 | 63 | 176.7739 | 64 | 179.6023 | 65 | 182.4308 |
| 66 | 185.2593 | 67 | 188.0877 | 68 | 190.9162 | 69 | 193.7447 | 70 | 196.5731 |

| | | | | | | | | | |
|-----|----------|-----|----------|-----|----------|-----|----------|-----------|------------------------|
| 71 | 199.4016 | 72 | 202.2301 | 73 | 205.0585 | 74 | 207.8870 | 75 | 210.7155 |
| 76 | 213.5439 | 77 | 216.3724 | 78 | 219.2008 | 79 | 222.0293 | 80 | 224.8577 |
| 81 | 227.6862 | 82 | 230.5146 | 83 | 233.3431 | 84 | 236.1716 | 85 | <u>239.0000</u> |
| 86 | 241.8284 | 87 | 244.6569 | 88 | 247.4854 | 89 | 250.3138 | 90 | 253.1423 |
| 91 | 255.9707 | 92 | 258.7992 | 93 | 261.6276 | 94 | 264.4561 | 95 | 267.2845 |
| 96 | 270.1129 | 97 | 272.9414 | 98 | 275.7698 | 99 | 278.5983 | 100 | 281.4267 |
| 101 | 284.2552 | 102 | 287.0836 | 103 | 289.9120 | 104 | 292.7405 | 105 | 295.5689 |
| 106 | 298.3974 | 107 | 301.2258 | 108 | 304.0543 | 109 | 306.8827 | 110 | 309.7112 |
| 111 | 312.5396 | 112 | 315.3680 | 113 | 318.1965 | 114 | 321.0249 | 115 | 323.8534 |
| 116 | 326.6818 | 117 | 329.5103 | 118 | 332.3387 | 119 | 335.1671 | 120 | 337.9956 |
| 121 | 340.8240 | 122 | 343.6524 | 123 | 346.4809 | 124 | 349.3093 | 125 | 352.1378 |
| 126 | 354.9662 | 127 | 357.7946 | 128 | 360.6231 | 129 | 363.4515 | 130 | 366.2799 |
| 131 | 369.1084 | 132 | 371.9368 | 133 | 374.7653 | 134 | 377.5937 | 135 | 380.4221 |
| 136 | 383.2506 | 137 | 386.0790 | 138 | 388.9074 | 139 | 391.7359 | 140 | 394.5643 |
| 141 | 397.3928 | 142 | 400.2212 | 143 | 403.0496 | 144 | 405.8781 | 145 | 408.7065 |
| 146 | 411.5349 | 147 | 414.3634 | 148 | 417.1918 | 149 | 420.0202 | 150 | 422.8487 |
| 151 | 425.6771 | 152 | 428.5056 | 153 | 431.3340 | 154 | 434.1624 | 155 | 436.9908 |
| 156 | 439.8193 | 157 | 442.6477 | 158 | 445.4761 | 159 | 448.3046 | 160 | 451.1330 |
| 161 | 453.9615 | 162 | 456.7899 | 163 | 459.6183 | 164 | 462.4467 | 165 | 465.2752 |
| 166 | 468.1036 | 167 | 470.9321 | 168 | 473.7605 | 169 | 476.5889 | 170 | 479.4174 |
| 171 | 482.2458 | 172 | 485.0742 | 173 | 487.9026 | 174 | 490.7311 | 175 | 493.5595 |
| 176 | 496.3879 | 177 | 499.2164 | 178 | 502.0448 | 179 | 504.8733 | 180 | 507.7017 |
| 181 | 510.5301 | 182 | 513.3585 | 183 | 516.1870 | 184 | 519.0154 | 185 | 521.8439 |
| 186 | 524.6723 | 187 | 527.5007 | 188 | 530.3292 | 189 | 533.1576 | 190 | 535.9860 |
| 191 | 538.8145 | 192 | 541.6429 | 193 | 544.4713 | 194 | 547.2997 | 195 | 550.1282 |
| 196 | 552.9566 | 197 | 555.7850 | 198 | 558.6135 | 199 | 561.4419 | 200 | 564.2703 |
| 201 | 567.0988 | 202 | 569.9272 | 203 | 572.7556 | 204 | 575.5840 | 205 | 578.4125 |
| 206 | 581.2409 | 207 | 584.0693 | 208 | 586.8978 | 209 | 589.7262 | 210 | 592.5546 |
| 211 | 595.3831 | 212 | 598.2115 | 213 | 601.0399 | 214 | 603.8683 | 215 | 606.6968 |
| 216 | 609.5252 | 217 | 612.3536 | 218 | 615.1821 | 219 | 618.0105 | 220 | 620.8389 |
| 221 | 623.6674 | 222 | 626.4958 | 223 | 629.3242 | 224 | 632.1526 | 225 | 634.9811 |
| 226 | 637.8095 | 227 | 640.6379 | 228 | 643.4664 | 229 | 646.2948 | 230 | 649.1232 |
| 231 | 651.9517 | 232 | 654.7801 | 233 | 657.6085 | 234 | 660.4370 | 235 | 663.2654 |
| 236 | 666.0938 | 237 | 668.9222 | 238 | 671.7507 | 239 | 674.5791 | 240 | 677.4075 |
| 241 | 680.2360 | 242 | 683.0644 | 243 | 685.8928 | 244 | 688.7213 | 245 | 691.5497 |
| 246 | 694.3781 | 247 | 697.2065 | 248 | 700.0350 | 249 | 702.8634 | 250 | 705.6918 |
| 251 | 708.5203 | 252 | 711.3487 | 253 | 714.1771 | 254 | 717.0056 | 255 | 719.8340 |
| 256 | 722.6624 | 257 | 725.4908 | 258 | 728.3193 | 259 | 731.1477 | 260 | 733.9761 |
| 261 | 736.8046 | 262 | 739.6330 | 263 | 742.4614 | 264 | 745.2899 | 265 | 748.1183 |
| 266 | 750.9467 | 267 | 753.7751 | 268 | 756.6036 | 269 | 759.4320 | 270 | 762.2604 |

| | | | | | | | |
|-----|-----------|-----|-----------|-----|-----------|-----|-----------|
| 271 | 765.0889 | 272 | 767.9173 | 273 | 770.7457 | 274 | 73.5742 |
| 275 | 776.4026 | 276 | 779.2310 | 277 | 782.0594 | 278 | 784.8879 |
| 279 | 787.7163 | 280 | 790.5447 | 281 | 793.3732 | 282 | 796.2016 |
| 283 | 799.0300 | 284 | 801.8585 | 285 | 804.6869 | 286 | 807.5153 |
| 287 | 810.3438 | 288 | 813.1722 | 289 | 816.0006 | 290 | 818.8290 |
| 291 | 821.6575 | 292 | 824.4859 | 293 | 827.3143 | 294 | 830.1428 |
| 295 | 832.9712 | 296 | 835.7996 | 297 | 838.6281 | 298 | 841.4565 |
| 299 | 844.2849 | 300 | 847.1133 | 301 | 849.9418 | 302 | 852.7702 |
| 303 | 855.5986 | 304 | 858.4271 | 305 | 861.2555 | 306 | 864.0839 |
| 307 | 866.9124 | 308 | 869.7408 | 309 | 872.5692 | 310 | 875.3976 |
| 311 | 878.2261 | 312 | 881.0545 | 313 | 883.8829 | 314 | 886.7114 |
| 315 | 889.5398 | 316 | 892.3682 | 317 | 895.1967 | 318 | 898.0251 |
| 319 | 900.8535 | 320 | 903.6819 | 321 | 906.5103 | 322 | 909.3387 |
| 323 | 912.1672 | 324 | 914.9956 | 325 | 917.8240 | 326 | 920.6525 |
| 327 | 923.4809 | 328 | 926.3093 | 329 | 929.1378 | 330 | 931.9662 |
| 331 | 934.7946 | 332 | 937.6230 | 333 | 940.4515 | 334 | 943.2799 |
| 335 | 946.1083 | 336 | 948.9368 | 337 | 951.7652 | 338 | 954.5936 |
| 339 | 957.4221 | 340 | 960.2505 | 341 | 963.0789 | 342 | 965.9073 |
| 343 | 968.7358 | 344 | 971.5642 | 345 | 974.3926 | 346 | 977.2211 |
| 347 | 980.0495 | 348 | 982.8779 | 349 | 985.7064 | 350 | 988.5348 |
| 351 | 991.3632 | 352 | 994.1917 | 353 | 997.0201 | 354 | 999.8485 |
| 355 | 1002.6769 | 356 | 1005.5054 | 357 | 1008.3338 | 358 | 1011.1622 |
| 359 | 1013.9907 | 360 | 1016.8190 | 361 | 1019.6475 | 362 | 1022.4759 |
| 363 | 1025.3043 | 364 | 1028.1328 | 365 | 1030.9612 | 366 | 1033.7897 |
| 367 | 1036.6180 | 368 | 1039.4465 | 369 | 1042.2749 | 370 | 1045.1034 |
| 371 | 1047.9318 | 372 | 1050.7603 | 373 | 1053.5886 | 374 | 1056.4171 |
| 375 | 1059.2455 | 376 | 1062.0740 | 377 | 1064.9023 | 378 | 1067.7307 |
| 379 | 1070.5592 | 380 | 1073.3876 | 381 | 1076.2161 | 382 | 1079.0444 |
| 383 | 1081.8729 | 384 | 1084.7013 | 385 | 1087.5298 | 386 | 1090.3582 |
| 387 | 1093.1866 | 388 | 1096.0150 | 389 | 1098.8435 | 390 | 1101.6719 |
| 391 | 1104.5004 | 392 | 1107.3287 | 393 | 1110.1572 | 394 | 1112.9856 |
| 395 | 1115.8141 | 396 | 1118.6425 | 397 | 1121.4709 | 398 | 1124.2993 |
| 399 | 1127.1278 | 400 | 1129.9562 | 401 | 1132.7847 | 402 | 1135.6130 |
| 403 | 1138.4415 | 404 | 1141.2699 | 405 | 1144.0984 | 406 | 1146.9268 |
| 407 | 1149.7552 | 408 | 1152.5836 | 409 | 1155.4120 | 410 | 1158.2405 |
| 411 | 1161.0688 | 412 | 1163.8973 | 413 | 1166.7257 | 414 | 1169.5542 |
| 415 | 1172.3826 | 416 | 1175.2111 | 417 | 1178.0394 | 418 | 1180.8679 |
| 419 | 1183.6963 | 420 | 1186.5248 | 421 | 1189.3531 | 422 | 1192.1816 |
| 423 | 1195.0100 | 424 | 1197.8385 | 425 | 1200.6669 | 426 | 1203.4954 |
| 427 | 1206.3237 | 428 | 1209.1522 | 429 | 1211.9806 | 430 | 1214.8091 |

| | | | | | | | |
|-----|-----------|-----|-----------|------------|------------------|-----|-----------|
| 431 | 1217.6375 | 432 | 1220.4659 | 433 | 1223.2943 | 434 | 1226.1228 |
| 435 | 1228.9512 | 436 | 1231.7797 | 437 | 1234.6080 | 438 | 1237.4365 |
| 439 | 1240.2649 | 440 | 1243.0933 | 441 | 1245.9218 | 442 | 1248.7501 |
| 443 | 1251.5786 | 444 | 1254.4070 | 445 | 1257.2355 | 446 | 1260.0638 |
| 447 | 1262.8923 | 448 | 1265.7207 | 449 | 1268.5492 | 450 | 1271.3776 |
| 451 | 1274.2061 | 452 | 1277.0344 | 453 | 1279.8629 | 454 | 1282.6913 |
| 455 | 1285.5198 | 456 | 1288.3481 | 457 | 1291.1766 | 458 | 1294.0050 |
| 459 | 1296.8335 | 460 | 1299.6619 | 461 | 1302.4904 | 462 | 1305.3187 |
| 463 | 1308.1472 | 464 | 1310.9756 | 465 | 1313.8041 | 466 | 1316.6324 |
| 467 | 1319.4608 | 468 | 1322.2893 | 469 | 1325.1177 | 470 | 1327.9462 |
| 471 | 1330.7745 | 472 | 1333.6030 | 473 | 1336.4314 | 474 | 1339.2599 |
| 475 | 1342.0883 | 476 | 1344.9167 | 477 | 1347.7451 | 478 | 1350.5736 |
| 479 | 1353.4020 | 480 | 1356.2305 | 481 | 1359.0588 | 482 | 1361.8873 |
| 483 | 1364.7157 | 484 | 1367.5442 | 485 | 1370.3726 | 486 | 1373.2010 |
| 487 | 1376.0294 | 488 | 1378.8579 | 489 | 1381.6863 | 490 | 1384.5148 |
| 491 | 1387.3431 | 492 | 1390.1716 | 493 | 1393.0000 | 494 | 1395.8284 |
| 495 | 1398.6569 | 496 | 1401.4852 | 497 | 1404.3137 | 498 | 1407.1421 |
| 499 | 1409.9706 | 500 | 1412.7990 | 501 | 1415.6274 | 502 | 1418.4558 |
| 503 | 1421.2843 | 504 | 1424.1127 | 505 | 1426.9412 | 506 | 1429.7695 |
| 507 | 1432.5980 | 508 | 1435.4264 | 509 | 1438.2549 | 510 | 1441.0833 |
| 511 | 1443.9117 | 512 | 1446.7401 | 513 | 1449.5686 | 514 | 1452.3970 |
| 515 | 1455.2255 | 516 | 1458.0538 | 517 | 1460.8823 | 518 | 1463.7107 |
| 519 | 1466.5391 | 520 | 1469.3676 | 521 | 1472.1959 | 522 | 1475.0244 |
| 523 | 1477.8528 | 524 | 1480.6813 | 525 | 1483.5096 | 526 | 1486.3381 |
| 527 | 1489.1665 | 528 | 1491.9950 | 529 | 1494.8234 | 530 | 1497.6519 |
| 531 | 1500.4802 | 532 | 1503.3087 | 533 | 1506.1371 | 534 | 1508.9656 |
| 535 | 1511.7939 | 536 | 1514.6224 | 537 | 1517.4508 | 538 | 1520.2793 |
| 539 | 1523.1077 | 540 | 1525.9362 | 541 | 1528.7645 | 542 | 1531.5930 |
| 543 | 1534.4214 | 544 | 1537.2499 | 545 | 1540.0782 | 546 | 1542.9066 |
| 547 | 1545.7351 | 548 | 1548.5635 | 549 | 1551.3920 | 550 | 1554.2203 |
| 551 | 1557.0488 | 552 | 1559.8772 | 553 | 1562.7057 | 554 | 1565.5341 |
| 555 | 1568.3625 | 556 | 1571.1909 | 557 | 1574.0194 | 558 | 1576.8478 |
| 559 | 1579.6763 | 560 | 1582.5046 | 561 | 1585.3331 | 562 | 1588.1615 |
| 563 | 1590.9900 | 564 | 1593.8184 | 565 | 1596.6469 | 566 | 1599.4752 |
| 567 | 1602.3037 | 568 | 1605.1321 | 569 | 1607.9604 | 570 | 1610.7889 |
| 571 | 1613.6173 | 572 | 1616.4458 | 573 | 1619.2742 | 574 | 1622.1027 |
| 575 | 1624.9310 | 576 | 1627.7595 | 577 | 1630.5879 | 578 | 1633.4164 |
| 579 | 1636.2448 | 580 | 1639.0732 | 581 | 1641.9016 | 582 | 1644.7301 |
| 583 | 1647.5585 | 584 | 1650.3870 | 585 | 1653.2153 | 586 | 1656.0438 |
| 587 | 1658.8722 | 588 | 1661.7007 | 589 | 1664.5291 | 590 | 1667.3575 |

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|-----|-----------|-----|-----------|-----|-----------|-----|-----------|
| 591 | 1670.1859 | 592 | 1673.0144 | 593 | 1675.8428 | 594 | 1678.6711 |
| 595 | 1681.4996 | 596 | 1684.3280 | 597 | 1687.1565 | 598 | 1689.9849 |
| 599 | 1692.8134 | 600 | 1695.6417 | 601 | 1698.4702 | 602 | 1701.2986 |
| 603 | 1704.1271 | 604 | 1706.9554 | 605 | 1709.7839 | 606 | 1712.6123 |
| 607 | 1715.4408 | 608 | 1718.2692 | 609 | 1721.0977 | 610 | 1723.9260 |
| 611 | 1726.7545 | 612 | 1729.5829 | 613 | 1732.4114 | 614 | 1735.2397 |
| 615 | 1738.0682 | 616 | 1740.8966 | 617 | 1743.7250 | 618 | 1746.5535 |
| 619 | 1749.3818 | 620 | 1752.2103 | 621 | 1755.0387 | 622 | 1757.8672 |
| 623 | 1760.6956 | 624 | 1763.5240 | 625 | 1766.3524 | 626 | 1769.1809 |
| 627 | 1772.0093 | 628 | 1774.8378 | 629 | 1777.6661 | 630 | 1780.4946 |
| 631 | 1783.3230 | 632 | 1786.1515 | 633 | 1788.9799 | 634 | 1791.8083 |
| 635 | 1794.6367 | 636 | 1797.4652 | 637 | 1800.2936 | 638 | 1803.1221 |
| 639 | 1805.9504 | 640 | 1808.7789 | 641 | 1811.6073 | 642 | 1814.4357 |
| 643 | 1817.2642 | 644 | 1820.0925 | 645 | 1822.9210 | 646 | 1825.7494 |
| 647 | 1828.5779 | 648 | 1831.4063 | 649 | 1834.2347 | 650 | 1837.0631 |
| 651 | 1839.8916 | 652 | 1842.7200 | 653 | 1845.5485 | 654 | 1848.3768 |
| 655 | 1851.2053 | 656 | 1854.0337 | 657 | 1856.8622 | 658 | 1859.6906 |
| 659 | 1862.5190 | 660 | 1865.3474 | 661 | 1868.1759 | 662 | 1871.0043 |
| 663 | 1873.8328 | 664 | 1876.6611 | 665 | 1879.4895 | 666 | 1882.3180 |
| 667 | 1885.1464 | 668 | 1887.9749 | 669 | 1890.8032 | 670 | 1893.6317 |
| 671 | 1896.4601 | 672 | 1899.2886 | 673 | 1902.1169 | 674 | 1904.9454 |
| 675 | 1907.7738 | 676 | 1910.6023 | 677 | 1913.4307 | 678 | 1916.2592 |
| 679 | 1919.0875 | 680 | 1921.9160 | 681 | 1924.7444 | 682 | 1927.5729 |
| 683 | 1930.4012 | 684 | 1933.2297 | 685 | 1936.0581 | 686 | 1938.8866 |
| 687 | 1941.7150 | 688 | 1944.5433 | 689 | 1947.3718 | 690 | 1950.2002 |
| 691 | 1953.0287 | 692 | 1955.8571 | 693 | 1958.6855 | 694 | 1961.5139 |
| 695 | 1964.3424 | 696 | 1967.1708 | 697 | 1969.9993 | 698 | 1972.8276 |
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| 703 | 1986.9698 | 704 | 1989.7982 | 705 | 1992.6267 | 706 | 1995.4551 |
| 707 | 1998.2836 | 708 | 2001.1119 | 709 | 2003.9403 | 710 | 2006.7688 |
| 711 | 2009.5972 | 712 | 2012.4257 | 713 | 2015.2540 | 714 | 2018.0825 |
| 715 | 2020.9109 | 716 | 2023.7394 | 717 | 2026.5677 | 718 | 2029.3962 |
| 719 | 2032.2246 | 720 | 2035.0531 | 721 | 2037.8815 | 722 | 2040.7100 |
| 723 | 2043.5383 | 724 | 2046.3668 | 725 | 2049.1953 | 726 | 2052.0237 |
| 727 | 2054.8521 | 728 | 2057.6804 | 729 | 2060.5090 | 730 | 2063.3374 |
| 731 | 2066.1658 | 732 | 2068.9941 | 733 | 2071.8225 | 734 | 2074.6511 |
| 735 | 2077.4795 | 736 | 2080.3079 | 737 | 2083.1362 | 738 | 2085.9648 |
| 739 | 2088.7932 | 740 | 2091.6216 | 741 | 2094.4500 | 742 | 2097.2786 |
| 743 | 2100.1069 | 744 | 2102.9353 | 745 | 2105.7637 | 746 | 2108.5923 |
| 747 | 2111.4207 | 748 | 2114.2490 | 749 | 2117.0774 | 750 | 2119.9060 |

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|-----|-----------|-----|-----------|-----|-----------|-----|-----------|
| 751 | 2122.7344 | 752 | 2125.5627 | 753 | 2128.3911 | 754 | 2131.2195 |
| 755 | 2134.0481 | 756 | 2136.8765 | 757 | 2139.7048 | 758 | 2142.5332 |
| 759 | 2145.3618 | 760 | 2148.1902 | 761 | 2151.0186 | 762 | 2153.8469 |
| 763 | 2156.6755 | 764 | 2159.5039 | 765 | 2162.3323 | 766 | 2165.1606 |
| 767 | 2167.9893 | 768 | 2170.8176 | 769 | 2173.6460 | 770 | 2176.4744 |
| 771 | 2179.3030 | 772 | 2182.1313 | 773 | 2184.9597 | 774 | 2187.7881 |
| 775 | 2190.6167 | 776 | 2193.4451 | 777 | 2196.2734 | 778 | 2199.1018 |
| 779 | 2201.9302 | 780 | 2204.7588 | 781 | 2207.5872 | 782 | 2210.4155 |
| 783 | 2213.2439 | 784 | 2216.0725 | 785 | 2218.9009 | 786 | 2221.7292 |
| 787 | 2224.5576 | 788 | 2227.3862 | 789 | 2230.2146 | 790 | 2233.0430 |
| 791 | 2235.8713 | 792 | 2238.7000 | 793 | 2241.5283 | 794 | 2244.3567 |
| 795 | 2247.1851 | 796 | 2250.0137 | 797 | 2252.8420 | 798 | 2255.6704 |
| 799 | 2258.4988 | 800 | 2261.3271 | 801 | 2264.1558 | 802 | 2266.9841 |
| 803 | 2269.8125 | 804 | 2272.6409 | 805 | 2275.4695 | 806 | 2278.2979 |
| 807 | 2281.1262 | 808 | 2283.9546 | 809 | 2286.7832 | 810 | 2289.6116 |
| 811 | 2292.4399 | 812 | 2295.2683 | 813 | 2298.0969 | 814 | 2300.9253 |
| 815 | 2303.7537 | 816 | 2306.5820 | 817 | 2309.4106 | 818 | 2312.2390 |
| 819 | 2315.0674 | 820 | 2317.8958 | 821 | 2320.7241 | 822 | 2323.5527 |
| 823 | 2326.3811 | 824 | 2329.2095 | 825 | 2332.0378 | 826 | 2334.8665 |
| 827 | 2337.6948 | 828 | 2340.5232 | 829 | 2343.3516 | 830 | 2346.1802 |
| 831 | 2349.0085 | 832 | 2351.8369 | 833 | 2354.6653 | 834 | 2357.4939 |
| 835 | 2360.3223 | 836 | 2363.1506 | 837 | 2365.9790 | 838 | 2368.8076 |
| 839 | 2371.6360 | 840 | 2374.4644 | 841 | 2377.2927 | 842 | 2380.1211 |
| 843 | 2382.9497 | 844 | 2385.7781 | 845 | 2388.6064 | 846 | 2391.4348 |
| 847 | 2394.2634 | 848 | 2397.0918 | 849 | 2399.9202 | 850 | 2402.7485 |
| 851 | 2405.5771 | 852 | 2408.4055 | 853 | 2411.2339 | 854 | 2414.0623 |
| 855 | 2416.8909 | 856 | 2419.7192 | 857 | 2422.5476 | 858 | 2425.3760 |
| 859 | 2428.2046 | 860 | 2431.0330 | 861 | 2433.8613 | 862 | 2436.6897 |
| 863 | 2439.5183 | 864 | 2442.3467 | 865 | 2445.1750 | 866 | 2448.0034 |
| 867 | 2450.8318 | 868 | 2453.6604 | 869 | 2456.4888 | 870 | 2459.3171 |
| 871 | 2462.1455 | 872 | 2464.9741 | 873 | 2467.8025 | 874 | 2470.6309 |
| 875 | 2473.4592 | 876 | 2476.2878 | 877 | 2479.1162 | 878 | 2481.9446 |
| 879 | 2484.7729 | 880 | 2487.6016 | 881 | 2490.4299 | 882 | 2493.2583 |
| 883 | 2496.0867 | 884 | 2498.9153 | 885 | 2501.7437 | 886 | 2504.5720 |
| 887 | 2507.4004 | 888 | 2510.2288 | 889 | 2513.0574 | 890 | 2515.8857 |
| 891 | 2518.7141 | 892 | 2521.5425 | 893 | 2524.3711 | 894 | 2527.1995 |
| 895 | 2530.0278 | 896 | 2532.8562 | 897 | 2535.6848 | 898 | 2538.5132 |
| 899 | 2541.3416 | 900 | 2544.1699 | 901 | 2546.9985 | 902 | 2549.8269 |
| 903 | 2552.6553 | 904 | 2555.4836 | 905 | 2558.3123 | 906 | 2561.1406 |
| 907 | 2563.9690 | 908 | 2566.7974 | 909 | 2569.6257 | 910 | 2572.4543 |

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|-----|-----------|------|-----------|-----|-----------|-----|-----------|
| 911 | 2575.2827 | 912 | 2578.1111 | 913 | 2580.9395 | 914 | 2583.7681 |
| 915 | 2586.5964 | 916 | 2589.4248 | 917 | 2592.2532 | 918 | 2595.0818 |
| 919 | 2597.9102 | 920 | 2600.7385 | 921 | 2603.5669 | 922 | 2606.3955 |
| 923 | 2609.2239 | 924 | 2612.0522 | 925 | 2614.8806 | 926 | 2617.7092 |
| 927 | 2620.5376 | 928 | 2623.3660 | 929 | 2626.1943 | 930 | 2629.0227 |
| 931 | 2631.8513 | 932 | 2634.6797 | 933 | 2637.5081 | 934 | 2640.3364 |
| 935 | 2643.1650 | 936 | 2645.9934 | 937 | 2648.8218 | 938 | 2651.6501 |
| 939 | 2654.4788 | 940 | 2657.3071 | 941 | 2660.1355 | 942 | 2662.9639 |
| 943 | 2665.7925 | 944 | 2668.6208 | 945 | 2671.4492 | 946 | 2674.2776 |
| 947 | 2677.1062 | 948 | 2679.9346 | 949 | 2682.7629 | 950 | 2685.5913 |
| 951 | 2688.4197 | 952 | 2691.2483 | 953 | 2694.0767 | 954 | 2696.9050 |
| 955 | 2699.7334 | 956 | 2702.5620 | 957 | 2705.3904 | 958 | 2708.2188 |
| 959 | 2711.0471 | 960 | 2713.8757 | 961 | 2716.7041 | 962 | 2719.5325 |
| 963 | 2722.3608 | 964 | 2725.1895 | 965 | 2728.0178 | 966 | 2730.8462 |
| 967 | 2733.6746 | 968 | 2736.5032 | 969 | 2739.3315 | 970 | 2742.1599 |
| 971 | 2744.9883 | 972 | 2747.8167 | 973 | 2750.6453 | 974 | 2753.4736 |
| 975 | 2756.3020 | 976 | 2759.1304 | 977 | 2761.9590 | 978 | 2764.7874 |
| 979 | 2767.6157 | 980 | 2770.4441 | 981 | 2773.2727 | 982 | 2776.1011 |
| 983 | 2778.9294 | 984 | 2781.7578 | 985 | 2784.5864 | 986 | 2787.4148 |
| 987 | 2790.2432 | 988 | 2793.0715 | 989 | 2795.9001 | 990 | 2798.7285 |
| 991 | 2801.5569 | 992 | 2804.3853 | 993 | 2807.2136 | 994 | 2810.0422 |
| 995 | 2812.8706 | 996 | 2815.6990 | 997 | 2818.5273 | 998 | 2821.3560 |
| 999 | 2824.1843 | 1000 | 2827.0127 | | | | |

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Simple closed null curves in Minkowski 3-Space

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Abstract In this work, we present some characterizations of simple closed null curves in Minkowski 3-space. Moreover, in the same space, we study a special case of such curves “Null curves of constant breadth”.

Keywords Minkowski 3-Space, simple closed null curve, curves of constant breadth.

§1. Introduction

At the beginning of the twentieth century, A. Einstein’s theory opened a door to new geometries such as Minkowski space, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold.

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifolds. For instance, in [6] and [10], the authors studied space-like and time-like curves of constant breadth in Minkowski 3-space, respectively. These studies have been treated based of the papers [1], [3], [8] and [9], in the spaces E^2 , E^3 and E^4 .

In this work, we investigate position vector of a simple closed null curve and give some characterizations in Minkowski space E_1^3 . Additionally, we express a characterization in the case of constant breadth.

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented (A more complete elementary treatment can be found in [2]).

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3 . Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors φ' are respectively space-like, time-like or null (light-like), for every $s \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|g(a, a)|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v, w) = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3 . For an arbitrary curve φ with first and second curvature, κ and τ in the space E_1^3 , the following Frenet formulae are given in [4]:

If φ is a null curve, then the Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \tau & 0 & -\kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (1)$$

satisfying equations

$$\begin{aligned} g(T, T) = g(B, B) = g(T, N) = g(N, B) = 0, \\ g(N, N) = 1, g(T, B) = 1. \end{aligned}$$

In this case, κ can take only two values: $\kappa = 0$ when α is a null straight line, or $\kappa = 1$ in all other cases.

Recall that an arbitrary curve is called a W -curve if it has constant Frenet curvatures [7]. In the rest of the paper, we shall assume $\kappa = 1$ at every point.

§3. Simple closed null curves in Minkowski 3-Space

Let $\varphi = \varphi(s)$ be null curve in the space E_1^3 . Moreover, let us suppose $\varphi = \varphi(s)$ simple closed null curve in the space E_1^3 . These curves will be denoted by (C) . The normal plane at every point P on the curve meets the curve at a single point Q other than P . We call the point Q the opposite point of P . We consider a curve in the class Γ as in [5] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. Here, we suppose both φ and φ^* are unit speed null curves of E_1^3 . A simple closed null curve having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^* = \varphi + \lambda T + \mu N + \delta B. \quad (2)$$

where λ, μ and δ are arbitrary functions of s and φ and φ^* are opposite points. Differentiating both sides of (2) and considering Frenet equations, we have

$$\frac{d\varphi^*}{ds} = T^* = \left\{ \frac{d\lambda}{ds} + \mu\tau + 1 \right\} T + \left\{ \frac{d\mu}{ds} + \lambda - \delta\tau \right\} N + \left\{ \frac{d\delta}{ds} - \mu \right\} B. \quad (3)$$

We know that $T^* = -T$. Then, we get the following system of ordinary differential equations:

$$\begin{aligned} \lambda' &= -\mu\tau - 2 \\ \mu' &= \delta\tau - \lambda \\ \delta' &= \mu \end{aligned} \quad (4)$$

Using system (4), we have the following differential equation with respect to λ as

$$\frac{d}{ds} \left\{ \frac{1}{\tau} \frac{d}{ds} \left[\frac{1}{\tau} \left(\frac{d\lambda}{ds} + 2 \right) \right] \right\} - \frac{d}{ds} \left(\frac{\lambda}{\tau} \right) - \frac{1}{\tau} \left(\frac{d\lambda}{ds} + 2 \right) = 0. \quad (5)$$

Theorem 3.1. This obtained differential equation of third order with variable coefficients (5) is a characterization for the simple closed null curve in E_1^3 . Via its solution, position vector of φ^* can be determined.

However, a general solution of (5) has not yet been found. Let us suppose $\varphi = \varphi(s)$ be a null W -curve in the space E_1^3 . In this case the differential equation (5) transforms to

$$\frac{d^3\lambda}{ds^3} - 2\tau \frac{d\lambda}{ds} - 2\tau = 0. \quad (6)$$

According to signature of τ , we study the following cases.

Case 3.1. $\tau > 0$. Then, we have the solution of the differential equation (6) as

$$\lambda = \Phi_1 \cosh \sqrt{2\tau}s + \Phi_2 \sinh \sqrt{2\tau}s - s \quad (7)$$

where $\Phi_1, \Phi_2 \in \mathbb{R}$. Thus, we may express other components, respectively,

$$\mu = -\sqrt{\frac{2}{\tau}} \left(\Phi_1 \sinh \sqrt{2\tau}s + \Phi_2 \cosh \sqrt{2\tau}s + \frac{1}{\sqrt{2\tau}} \right) \quad (8)$$

and

$$\delta = -\frac{1}{\tau} \left(\Phi_1 \cosh \sqrt{2\tau}s + \Phi_2 \sinh \sqrt{2\tau}s - s \right). \quad (9)$$

Case 3.2. $\tau < 0$. In this case, we have the solution

$$\lambda = \Psi_1 \cos \sqrt{-2\tau}s + \Psi_2 \sin \sqrt{-2\tau}s - s \quad (10)$$

for the real numbers $\Psi_1, \Psi_2 \in \mathbb{R}$. Since, we write other components

$$\mu = -\frac{1}{\tau} \sqrt{-2\tau} \left(-\Phi_1 \sin \sqrt{-2\tau}s + \Phi_2 \cos \sqrt{-2\tau}s + \frac{1}{\sqrt{-2\tau}} \right) \quad (11)$$

and

$$\delta = -\frac{1}{\tau} \left(\Psi_1 \cos \sqrt{-2\tau}s + \Psi_2 \sin \sqrt{-2\tau}s - s \right). \quad (12)$$

Theorem 3.2. Position vector of a simple closed null W -curve in Minkowski space E_1^3 can be composed by the components (7), (8) and (9); or (10), (11) and (12) according to signature of the torsion.

§4. Null curves of constant breadth in Minkowski 3-Space

In this section, we give a characterization of the null curves of constant breadth in Minkowski 3-space.

Let us suppose the null curve treated in the previous section has constant breadth. In another words, let the distance between opposite points of C and C^* be constant. Then, due to null frame vectors, we may express

$$\|\varphi^* - \varphi\| = 2\lambda\delta + \mu^2 = l^2 = \text{constant}. \quad (13)$$

Differentiating both sides of (13), we immediately arrive

$$\delta \frac{d\lambda}{ds} + \lambda \frac{d\delta}{ds} + \mu \frac{d\mu}{ds} = 0. \quad (14)$$

Considering system of ordinary differential equations (4), we have

$$\delta = 0 \quad (15)$$

and, thereafter

$$\lambda = \mu = 0. \quad (16)$$

Since, we give:

Theorem 4.1. There does not exist null curves of constant breadth in Minkowski 3-space E_1^3 .

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On the Elliptic cylindrical Tzitzeica curves in Minkowski 3-Space

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Abstract Elliptic cylindrical curves satisfying Tzitzeica condition are obtained via the solution of the forced harmonic equation in Minkowski 3-Space. In addition, we have given the conditions to be of spacelike, timelike and null curve of the elliptic cylindrical Tzitzeica curve.

Keywords Tzitzeica curve, Elliptic cylindrical curve, Minkowski 3-Space.

§1. Preliminaries and Introduction

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the Lorentzian inner product

$$\langle x, y \rangle_L = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. An arbitrary vector $x = (x_1, x_2, x_3)$ in E_1^3 can have one of three Lorentzian causal characters: it is spacelike if $\langle x, x \rangle_L > 0$ or $x = 0$, timelike if $\langle x, x \rangle_L < 0$ and null (lightlike) if $\langle x, x \rangle_L = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors (tangents) $\alpha'(s) = T(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. Lorentzian vectorial product of x and y is defined by

$$x \wedge_L y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2).$$

Recall that the pseudo-norm of an arbitrary vector $x \in E_1^3$ is given by $\|x\|_L = \sqrt{|\langle x, x \rangle_L|}$. If the curve α is non-unit speed, then

$$\kappa(t) = \frac{\|\alpha'(t) \wedge_L \alpha''(t)\|_L}{\|\alpha'(t)\|_L^3}, \tau(t) = \frac{\det(\alpha'(t), \alpha''(t), \alpha'''(t))}{\|\alpha'(t) \wedge_L \alpha''(t)\|_L^2}. \quad (1.1)$$

If the curve α is unit speed, then

$$\kappa(s) = \|\alpha''(s)\|_L, \tau(s) = \|B'(s)\|_L. \quad (1.2)$$

[1,4].

In this paper we have interested in Tzitzeica elliptic cylindrical curves in Minkowski 3-Space, more precisely we ask in what conditions a cylindrical curve is a Tzitzeica one, namely

the function $t \rightarrow \frac{\tau(t)}{d^2(t)}$ is constant, where $d(t)$ is the distance from origin to the osculating plane of curve. The Tzitzeica condition yields a third-order ODE which in our framework admits a direct integration. Therefore the final answer of main problem is given via a second order ODE which in the elliptic case is exactly the equation of a forced harmonic oscillator. In this case, the solution depends of four real constants: one defining the Tzitzeica condition and other three obtained by integration.

§2. Elliptic Cylindrical Tzitzeica curves in Minkowski 3-Space

Proposition 1. Let $\alpha(t)$ be an elliptic cylindrical curve in Minkowski 3-Space. Then, the curve $\alpha(t)$ is Tzitzeica curve if and only if

$$f(t) = f(0) \cos t + f'(0) \sin t + \int_0^t \frac{\sin(u-t)}{Ku+c} du,$$

where $f(0)$, $f'(0)$, $K \neq 0$ and c are real constants.

Proof. Let in E_1^3 a curve C given in vectorial form $C : \alpha = \alpha(t)$. This curve is called elliptic cylindrical if has the expression

$$\alpha(t) = (\cos t, \sin t, f(t)) \quad (2.1)$$

and differentiation of $\alpha(t)$, we have

$$\begin{aligned} \alpha'(t) &= (-\sin t, \cos t, f'(t)) \\ \alpha''(t) &= (-\cos t, -\sin t, f''(t)) \\ \alpha'''(t) &= (\sin t, -\cos t, f'''(t)) \end{aligned} \quad (2.2)$$

for some $f \in C^\infty(R)$. From Eq.(1.1) and Eq.(2.2), the torsion function is

$$\tau(t) = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge_L \alpha''\|_L^2} = \frac{f' + f'''}{f'^2 + f''^2 - 1}.$$

Then the distance from origin to the osculating plane is

$$d(t) = \frac{|f + f''|}{\sqrt{|f'^2 + f''^2 - 1|}}.$$

Let us suppose that the curve is Tzitzeica with the constant $K \neq 0$, because the curve is not contained in a plane

$$\begin{aligned} K(t) &= \frac{\tau(t)}{d^2(t)} \\ &= \frac{f' + f'''}{f'^2 + f''^2 - 1} \frac{|f'^2 + f''^2 - 1|^2}{|f + f''|^2} \\ &= \frac{f'(t) + f'''(t)}{(f(t) + f''(t))^2}. \end{aligned}$$

Integration gives

$$f''(t) + f(t) = -\frac{1}{Kt + c}, \quad (2.3)$$

where c is a real constant. Then the Laplace transform gives

$$[s^2 Y(s) - sf(0) - f'(0)] + Y(s) = -L\{G(t)\} = -g(s),$$

where $Y(s)$ and $G(s)$ denote the Laplace transform of $f(t)$ and $g(t)$ respectively, and $f(0)$ and $f'(0)$ are arbitrary constants. Hence

$$\begin{aligned} Y(s) &= f(0) \frac{s}{s^2 + 1} + f'(0) \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1} L\{G(t)\} \\ &= f(0) \frac{s}{s^2 + 1} + f'(0) \frac{1}{s^2 + 1} - \frac{1}{s^2 + 1} g(s) \end{aligned}$$

and therefore

$$f(t) = f(0) \cos t + f'(0) \sin t - \sin(t) * G(t),$$

where the function denoted by $\sin(t) * G(t)$ and defined as

$$\sin(t) * G(t) = \int_0^t G(u) \sin(u - t) du$$

is called the convolution of the functions $\sin t$ and $G(t)$ or

$$f(t) = f(0) \cos t + f'(0) \sin t + \int_0^t G(u) \sin(u - t) du.$$

Theorem 1. Let $\alpha(t)$ be elliptic cylindrical Tzitzeica curve, then the curve $\alpha(t)$ are space-like, timelike and null curve if and only if $f'^2(0) < 1$, $f'^2(0) > 1$ and $f'^2(0) = 1$, respectively.

Proof. Since the curve $\alpha(t) = (\cos t, \sin t, f(t))$, the tangent of the curve is $T = \alpha'(t) = (-\sin t, \cos t, f'(t))$. The taylor series of the function f in the neighbourhood of zero is

$$f(t) = f(0) - f'(0)t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \dots$$

We take into consideration satisfying

$$f(0) = f'(0) \neq 0, \quad 0 = f''(0) = f'''(0) = \dots \quad (2.4)$$

in the neighbourhood of zero. Then we have

$$f(0) \cos t + f'(0) \sin t + \int_0^t \frac{\sin(u - t)}{Ku + c} du = f(0) + f'(0)t.$$

From the last equation, we get

$$\begin{aligned} f(0) \cos t &= f(0), \\ f'(0) \sin t &= f'(0)t, \\ \int_0^t \frac{\sin(u - t)}{Ku + c} du &= 0. \end{aligned}$$

Then, for $t \rightarrow 0$

$$\begin{aligned}\cos t &= 1, \\ \sin t &= t, \\ \lim_{t \rightarrow 0} \int_0^t \frac{\sin(u-t)}{Ku+c} du &= 0.\end{aligned}$$

Thus, satisfying Eq.(2.4) as $t \rightarrow 0$, $\sin t = t$ and the function f is written such as

$$f(t) = f'(0) \sin t.$$

If we take derivative of the last equation for t and square, we have

$$\begin{aligned}f'^2(t) &= f'^2(0) \cos^2 t, \\ \langle T(t), T(t) \rangle_L &= 1 - f'^2 > 0,\end{aligned}$$

or

$$\begin{aligned}(f')^2(t) &< 1, \\ (f')^2(0) \cos^2 t &< 1.\end{aligned}$$

Since $|\cos t| < 1$, from the last equation, we have

(i) The curve $\alpha(t)$ is spacelike curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 > 0.$$

Then, we have

$$(f')^2(0) < 1.$$

(ii) The Tzitzeica curve $\alpha(t)$ is timelike curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 < 0.$$

Then, we have

$$f'^2(0) > 1.$$

(iii) The Tzitzeica curve $\alpha(t)$ is null curve if and only if

$$\langle T(t), T(t) \rangle_L = 1 - f'^2 = 0.$$

Then, we have

$$f'^2(0) = 1.$$

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On n -Fréchet spaces

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Abstract In this paper, we introduce the notion of n -sequential closure operators and use these operators to construct classes of n -Fréchet spaces as generalizations of the class of Fréchet spaces. Then we get some necessary and sufficiency conditions for a sequential space to be a Fréchet space. These results are generalizations of the preceding one of W. C. Hong in [5].

Keywords Fréchet, n -Fréchet, sequential, sequential closure, n -sequential closure.

§1. Introduction and results

M. Venkataraman poses the following problem in [8].

Problem 1.1. Characterize “the class of topological spaces which can be specified completely by knowledge of their convergent sequences”.

It is well known and useful fact that every first-countable space falls into this class. To solve Problem 1.1, S. P. Franklin introduced notions of Fréchet spaces and sequential spaces ([1], [2] and [3]). Recently, W. C. Hong has investigated the relation between Fréchet spaces and sequential spaces under the sequential closure operator c_1 [4],

$$c_1(A) = \{x \in X : \text{there exists a sequence } \{x_n : n \in \mathbb{N}\} \subset A, x_n \rightarrow x\},$$

for all subset A of X .

Recall the following notions.

Definition 1.2. ([2], [7]) Let (X, c) be a topological space endowed with the closure operator c .

- (1) X is a Fréchet space, if for all $A \subset X$, $c(A) = c_1(A)$.
- (2) X is a sequential space, if for all $A \subset X$, $A = c(A)$ whenever $A = c_1(A)$.
- (3) X has countable tightness, if for all $A \subset X$, $x \in c(C)$ for some countable subset C of A whenever $x \in c(A)$.

From the fact that the closure operator of Fréchet spaces is specified completely by knowledge of their convergent sequences, it is natural to ask whether we can generalize the class of Fréchet spaces under this property. That is, we are interested by the following question.

Question 1.3. Is there a class of spaces which contains properly the class of Fréchet spaces such that the closure operator of spaces in this class is specified completely by knowledge of their convergent sequences?

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In this paper, we introduce the notion of n -sequential closure operators c_n , $n \in \mathbb{N}$, and use these operators to give classes of n -Fréchet spaces as generalizations of the class of Fréchet spaces. Then, we answer affirmatively Question 1.3. As the application, we give necessary and sufficiency conditions for a sequential space to be a Fréchet space. These results are generalizations of the preceding one of W. C. Hong in [5].

Assume that all spaces are Hausdorff, all maps are continuous and onto, \mathbb{N} denotes the set of all natural numbers, and ω denotes $\mathbb{N} \cup \{0\}$. For terms which are not defined here, please refer to [1].

§2. Fréchet spaces

In this section we recall some results on Fréchet spaces.

Proposition 2.1. ([1], Theorem 1.6.14). Every first-countable space is a Fréchet space and every Fréchet space is a sequential space.

Proposition 2.2. ([7]). Every sequential space has countable tightness.

Definition 2.3. ([1]). Let $f : X \longrightarrow Y$ be a map.

(1) f is a pseudo-open map, if $y \in \text{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X .

(2) f is a quotient map, if U open in Y whenever $f^{-1}(U)$ open in X .

Example 2.4. There exists a sequential space without being Fréchet.

Proof. Recall the Arens' space S_2 [6]. Let $T_0 = \{a_n : n \in \mathbb{N}\}$ be a sequence converging to $a \notin T_0$, and T_n , $n \in \mathbb{N}$, be a sequence converging to $b_n \notin T_n$. Put $T = \bigoplus_{n \in \mathbb{N}} (T_n \cup \{b_n\})$. Then $S_2 = \{a\} \cup (\bigcup_{n \in \omega} T_n)$ is a quotient space of $(T_0 \cup \{a\}) \oplus T$ by identifying each $b_n \in T$ to $a_n \in T_0$. We get that S_2 is a sequential space without being Fréchet.

Proposition 2.5. ([2], Proposition 2.1).

(1) Every subspace of a Fréchet space is a Fréchet space.

(2) The disjoint topological sum of any family of Fréchet spaces is a Fréchet space.

Proposition 2.6. ([2], Proposition 2.3). Let $f : X \longrightarrow Y$ be a quotient map from a Fréchet space X onto Y . Then f is pseudo-open if and only if Y is Fréchet.

Corollary 2.7. Fréchet spaces are preserved by pseudo-open maps, particularly, by closed or open maps.

Proposition 2.8. ([3], Proposition 7.2). A sequential space is Fréchet if and only if it is hereditarily sequential.

Example 2.9. ([3], Example 7.4). The product of two Fréchet spaces can be sequential without being Fréchet.

Proposition 2.10. ([5], Theorem 2). A sequential space X is Fréchet if and only if $c_1(A) = c(A)$ for every countable subset A of X .

Proposition 2.11. ([2], Proposition 2.4) Every Fréchet space is precisely the pseudo-open image of a topological sum of convergent sequences.

Corollary 2.12. The following are equivalent for a space X .

- (1) X is a Fréchet space,
- (2) X is a pseudo-open image of a locally compact metric space,
- (3) X is a pseudo-open image of a locally separable metric space,

- (4) X is a pseudo-open image of a metric space.

§3. n -Fréchet spaces

Definition 3.1. Let (X, c) be a topological space endowed with the closure operator c . A map c_n , $n \in \mathbb{N}$, $n \geq 2$, defines by induction on n :

$$c_n(A) = c_1(c_{n-1}(A)),$$

for all $A \subset X$ to be called an n -sequential closure operator. The sequential closure operator c_1 is also called an 1-sequential closure operator for convince.

Proposition 3.2. Let (X, c) be a topological space endowed with the closure operator c . Then the following hold for all $n \in \mathbb{N}$.

- (1) $c_n(\emptyset) = \emptyset$.
- (2) $A \subset c_1(A) \subset \cdots \subset c_n(A) \subset c(A)$ for all $A \subset X$.
- (3) $c_n(A) \subset c_n(B)$ for all $A \subset B \subset X$.
- (4) $c_n(A \cup B) = c_n(A) \cup c_n(B)$ for all $A, B \subset X$.

Definition 3.3. Let (X, c) be a topological space endowed with the closure operator c , and $n \in \mathbb{N}$. Then X is called an n -Fréchet space, if $c(A) = c_n(A)$ for all $A \subset X$.

Remark 3.4. (1) An 1-Fréchet space is precisely a Fréchet space.

(2) By Proposition 3.2, $c_n(A) \subset c(A)$ for all $A \subset X$. Then, to prove that X is an n -Fréchet, it suffices to show that $c(A) \subset c_n(A)$ for all $A \subset X$.

Next, we construct i -spaces and i -sequences by induction on $i \in \mathbb{N}$ to consider the relation between n -Fréchet spaces.

Definition 3.5. Let X be a topological space.

(1) X is called an 1-space, if $X = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ with $x_n \rightarrow x \notin \{x_n : n \in \mathbb{N}\}$, and all x_n 's being distinct. Here x_n , $n \in \mathbb{N}$, is called a 0-limit point of X , and x is called an 1-limit point of X .

(2) For each $i \in \mathbb{N}$, $i \geq 2$, X is called an i -space, if $X = \{x\} \cup (\bigcup_{n \in \mathbb{N}} X_n)$ is a quotient space obtained from a topological sum $(X_0 \cup \{x\}) \oplus (\bigoplus_{n \in \mathbb{N}} (X_n \cup \{x_n\}))$ by identifying each $y_n \in X_n$ to $x_n \in X_0$, where X_n , $n \in \mathbb{N}$, is an $(i-1)$ -space with an $(i-1)$ -limit point y_n , and $X_0 = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is an 1-space with an 1-limit point x . Here x is called an i -limit point of X , and a j -limit point of X_n 's, $j = 0, \dots, i-1$, is called a j -limit point of X .

Remark 3.6. (1) Each 1-space is an infinite convergent sequence, and each 2-space is an Aren's space S_2 (see Example 2.4).

(2) For each $i \in \mathbb{N}$, an i -space is an infinite countable set.

Definition 3.7. Let X be a topological space and $S \subset X$.

(1) S is called an 1-sequence of X , if $S = \{x_n : n \in \mathbb{N}\}$ is a sequence converging to x . We say that S converges under 1 to x , and write $S \xrightarrow{1} x$, here x is called an 1-limit point of S .

(2) For $i \in \mathbb{N}$, $i \geq 2$, S is called an i -sequence of X , if $S = \bigcup_{n \in \mathbb{N}} L_n$, where L_n is an $(i-1)$ -sequence, $L_n \xrightarrow{i-1} x_n$ for each $n \in \mathbb{N}$, and $L_0 = \{x_n : n \in \mathbb{N}\}$ is an 1-sequence with $L_0 \xrightarrow{1} x$. We say that S converges under i to x , and write $S \xrightarrow{i} x$, here x is called an i -limit point of S .

Remark 3.8. The following hold.

- (1) The set S of all 0-limit points of an i -space X is an i -sequence satisfying that $c_i(S) = X$.
- (2) Each i -sequence is a countable set.
- (3) Every finite set is an i -sequence. But the following Example 3.9 proves that there is an infinite countable subset of a metric space without being an i -sequence.

Example 3.9. There is an infinite countable subset of a metric space without being an i -sequence.

Proof. Let \mathbb{R} be a discrete space. Then \mathbb{R} is a metric space. We have that \mathbb{N} is an infinite countable subset of \mathbb{R} , and \mathbb{N} is not an i -sequence. In fact, if there exists an $i \in \mathbb{N}$ such that \mathbb{N} is an i -sequence. Since each 1-sequence of \mathbb{R} with the discrete topology is finite, \mathbb{N} is finite. It is a contradiction.

Lemma 3.10. Let X be a topological space, and $A \subset X$. Then $x \in c_n(A)$ if and only if there exists an n -sequence $S \subset A$ such that $S \xrightarrow{n} x$.

Proof. Necessity. For $n = 1$, we get that $x \in c_1(A)$. Then there exists a sequence $S = \{x_n : n \in \mathbb{N}\} \subset A, x_n \rightarrow x$. It is clear that S is an 1-sequence, $S \subset A$, and $S \xrightarrow{1} x$.

For each $n \geq 2$, we get that $x \in c_n(A) = c_1(c_{(n-1)}(A))$. Then there exists an 1-sequence $L \subset c_{(n-1)}(A)$ such that $L \xrightarrow{1} x$. Put $L = \{x_i : i \in \mathbb{N}\}$ then $x_i \in c_{(n-1)}(A)$ for all $i \in \mathbb{N}$. Note that for each $i \in \mathbb{N}$, there is an $(n-1)$ -sequence $L_i \subset A$ such that $L_i \xrightarrow{n-1} x_i$. Then $S = \bigcup_{i \in \mathbb{N}} L_i \subset A$ is an n -sequence, and $S \xrightarrow{n} x$.

Sufficiency. Since $S \xrightarrow{n} x, x \in c_n(S)$. By Proposition 3.2, $x \in c_n(A)$

Corollary 3.11. Let X be an n -Fréchet space, and $A \subset X$. Then $x \in c(A)$ if and only if there exists an n -sequence $S \subset A$ such that $S \xrightarrow{n} x$.

Proposition 3.12. The following hold for a space X .

- (1) If X is an n -Fréchet space, then X is an $(n+1)$ -Fréchet space.
- (2) If X be an n -Fréchet space, then X is a sequential space.

Proof. (1) Let $A \subset X$ and $x \in c(A)$. Since X is n -Fréchet, $x \in c_n(A)$. Note that $c_n(A) \subset c_{(n+1)}(A)$. Then $x \in c_{(n+1)}(A)$. So $c(A) \subset c_{(n+1)}(A)$, i.e., X is an $(n+1)$ -Fréchet space.

(2) Let $A \subset X$ and $A = c_1(A)$. We need to prove that $A = c(A)$. It follows from $A = c_1(A)$ that $A = c_1(A) = \dots = c_{(n-1)}(A) = c_n(A)$. Since $c(A) = c_n(A)$, $A = c(A)$.

Corollary 3.13. Every n -Fréchet space has countable tightness.

The following examples to prove that inverse implications in Proposition 3.12 do not hold.

Example 3.14. For all $n \in \mathbb{N}$, there exists an $(n+1)$ -Fréchet space which is not an n -Fréchet space.

Proof. For each $n \in \mathbb{N}$, let X be an $(n+1)$ -space.

- (i) X is an $(n+1)$ -Fréchet space.

Let $A \subset X$ and $x \in c(A)$. We shall prove that $x \in c_{(n+1)}(A)$.

If x is a 0-limit point, then x is an isolated point. Then from $x \in c(A)$ we get $x \in A \subset c_{(n+1)}(A)$.

If x is an 1-limit point, then x has a countable neighborhood system. Then from $x \in c(A)$ we get that there exists a sequence $\{x_n : n \in \mathbb{N}\} \subset A$ such that $x_n \rightarrow x$. Hence $x \in c_1(A) \subset c_{(n+1)}(A)$.

If x is an i -limit point, $i = 2, \dots, n+1$, then there exist $(i-1)$ -spaces Y_m 's, $m \in \mathbb{N}$, with $(i-1)$ -limit points y_m 's, and an 1-sequence $Y_0 = \{y_m : m \in \mathbb{N}\}$ with the 1-limit point x such that $\{x\} \cup (\bigcup_{n \in \omega} Y_n) \subset X$, and $\{x\} \cup (\bigcup_{m \in \omega} Y_m)$ is an i -space. Since $x \in c(A)$, there is infinitely numbers $m \in \mathbb{N}$ such that $A \cap Y_m$ is an $(i-1)$ -sequence with the $(i-1)$ -limit point y_m . Then there is infinitely numbers m such that $y_m \in c_{(i-1)}(A \cap Y_m)$. Hence $x \in c_i(A \cap Y_m) \subset c_i(A)$. It implies that $x \in c_i(A) \subset c_{(n+1)}(A)$.

(ii) X is not an n -Fréchet space.

Let

$$A = \{x \in X : x \text{ is the } 0\text{-limit point of } X\}.$$

Then $c_n(A) = X - \{a\}$ with a is the $(n+1)$ -limit point of X . On the other hand, $a \in c(A)$. Therefore $c_n(A) \neq c(A)$. So X is not n -Fréchet.

Example 3.15. There is a sequential space which is not an n -Fréchet space for all $n \in \mathbb{N}$.

Proof. Let $S_\infty = \bigoplus_{m \in \mathbb{N}} S_m$, where S_m 's are m -spaces.

(i) S_∞ is a sequential space.

It follows from the proof of Example 3.14 that each m -space S_m is an m -Fréchet space. Then each S_m is a sequential space by Proposition 3.12. So S_∞ is a sequential space by ([2], Proposition 1.6).

(ii) S_∞ is not an n -Fréchet space.

For each $n \in \mathbb{N}$, we put

$$A = \{x \in S_{(n+1)} : x \text{ is the } 0\text{-limit point of } S_{(n+1)}\}.$$

Then $c_n(A) = S_{(n+1)} - \{x\}$, here x is the $(n+1)$ -limit point of $S_{(n+1)}$. On the other hand, $x \in c(A)$. Then $c(A) \neq c_n(A)$. It implies that S_∞ is not an n -Fréchet space.

Remark 3.16. It follows from Proposition 3.12, Example 3.14, and Example 3.15 that

$$\text{Fréchet} \Rightarrow n\text{-Fréchet} \Rightarrow (n+1)\text{-Fréchet} \Rightarrow \text{sequential},$$

for all $n \in \mathbb{N}$. Moreover, all of inverse implications do not hold if $n \geq 2$.

Lemma 3.17. Let $f : X \rightarrow Y$ be a map. If S is an n -sequence in X with the n -limit point x , then $f(S)$ is an n -sequence in Y with the n -limit point $f(x)$.

Proposition 3.18. Let $f : X \rightarrow Y$ be a map and X be an n -Fréchet space, then Y is an n -Fréchet space.

Proof. For all $A \subset Y$ and $y \in c(A)$, we shall prove that $y \in c_n(A)$. Suppose that $f^{-1}(y) \cap c(f^{-1}(A)) = \emptyset$. Then $U = X - c(f^{-1}(A))$ is an open set containing $f^{-1}(y)$. Since f is a pseudo-open map and $f^{-1}(y) \subset U$, $y \in \text{int}(f(U)) = \text{int}(f(X - c(f^{-1}(A)))) = \text{int}(Y - f(c(f^{-1}(A)))) \subset \text{int}(Y - A) = Y - c(A)$. So $y \notin c(A)$. It is a contradiction. Then there exists some point $x \in f^{-1}(y) \cap c(f^{-1}(A))$. Since X is an n -Fréchet space, there is an n -sequence $S \subset f^{-1}(A)$ such that $S \xrightarrow{n} x$ by Corollary 3.11. On the other hand, $y = f(x)$ is the n -limit point of n -sequence $f(S) \subset f(f^{-1}(A)) = A$ by Lemma 3.17. It implies that $y \in c_n(A)$.

Example 3.19. For all $n \geq 2$, there exists an n -Fréchet space without being hereditarily n -Fréchet.

Proof. For $n \geq 2$, let X be an n -space. Then X is an n -Fréchet space by the proof of Example 3.14. We shall prove that X is not hereditarily n -Fréchet. Conversely, suppose that X is hereditarily n -Fréchet. By Proposition 3.12, X is a hereditarily sequential space. It follows from Proposition 2.8 that X is an 1-Fréchet space. It is a contradiction (see Example 3.14). Then X is not hereditarily n -Fréchet.

Proposition 3.20. Every closed subspace of an n -Fréchet space is an n -Fréchet space.

Proof. Let X be an n -Fréchet space, $Y \subset X$, $Y = c(Y)$ and $A \subset Y$. Let c^Y and c_n^Y be the closure operator and the n -sequential closure operator of the subspace Y , respectively. We need to prove that $c^Y(A) = c_n^Y(A)$. Since $c_n(A) = c(A) \subset c(Y) = Y$, $c_n(A) = c_n^Y(A)$. On the other hand, $c^Y(A) = c(A) \cap Y = c(A) \cap c(Y) = c(A) = c_n(A)$. Then $c^Y(A) = c_n^Y(A)$.

Proposition 3.21. The disjoint topological sum of any family of n -Fréchet spaces is an n -Fréchet space.

Proof. Let $\{X_i : i \in I\}$ be a disjoint collection of n -Fréchet spaces, and $X = \bigoplus_{i \in I} X_i$. For any $A \subset X$ and $x \in c(A)$, we need to prove that $x \in c_n(A)$. Since $x \in c(A)$, there exists some $i \in I$ such that $x \in c(A) \cap X_i$. Then $x \in c^{X_i}(A) = c_n^{X_i}(A)$, here c^{X_i} and $c_n^{X_i}$ are the closure operator and n -sequential closure operator of X_i , respectively. Since $c_n^{X_i}(A) \subset c_n(A)$, $x \in c_n(A)$.

Proposition 3.22. A space having the countable tightness X is an n -Fréchet space if and only if $c(A) = c_n(A)$ for every countable subset $A \subset X$.

Proof. Necessary. It is clear.

Sufficiency. Let $c_n(A) = c(A)$ for every countable subset $A \subset X$. For any $B \subset X$ we need to prove that $c(B) = c_n(B)$. For any $x \in c(B)$, since X has the countable tightness, there is a countable subset $C \subset B$ such that $x \in c(C)$. Note that $c(C) = c_n(C)$ by countability of C . Then $x \in c_n(C) \subset c_n(B)$. It implies that $c(B) = c_n(B)$.

Corollary 3.23. The following hold for a space X .

- (1) Under X being a sequential space, then X is an n -Fréchet space if and only if $c(A) = c_n(A)$ for every countable subset $A \subset X$.
- (2) Under X being an m -Fréchet space and $m > n$, then X is an n -Fréchet space if and only if $c_m(A) = c_n(A)$ for every countable subset $A \subset X$.

Proof. It is straightforward from Proposition 2.2, Corollary 3.13, and Proposition 3.22.

Remark 3.24. For $n = 1$ in Corollary 3.23.(1) we get ([5], Theorem 2). Then, Proposition 3.22 is a generation of ([5], Theorem 2) (see Proposition 2.10).

Proposition 3.25. A sequential space X is an n -Fréchet space if and only if $c_n(S) = c_{(n+1)}(S)$ for every $(n+1)$ -sequence $S \subset X$.

Proof. Necessary. It is straightforward from Proposition 3.21.

Sufficiency. Conversely, suppose that X is not n -Fréchet. Then there exists $A \subset X$ such that $c_n(A) \neq c(A)$. Since $A \subset c_n(A) \subset c(A)$, $c(A) \subset c(c_n(A)) \subset c(c(A))$. Then $c(c_n(A)) = c(A)$. It implies that $c_n(A)$ is not closed in a sequential space X . So $c_n(A) \neq c_1(c_n(A))$. Hence there is some $x \in c_1(c_n(A))$ such that $x \notin c_n(A)$. Because $x \in c_1(c_n(A)) = c_{(n+1)}(A)$, there is an $(n+1)$ -sequence $S \subset A$ such that $S \xrightarrow{n+1} x$ by Lemma 3.10. Then $x \in c_{(n+1)}(S) = c_n(S) \subset c_n(A)$. It implies that $x \in c_n(A)$. It is a contradiction.

Corollary 3.26. A sequential space X is an 1-Fréchet space if and only if $c_1(S) = c_2(S)$

for every 2-sequence $S \subset X$.

Corollary 3.27. A space X is an 1-Fréchet space if and only if $c_1(S) = c(S)$ for every 2-sequence $S \subset X$.

Proof. Necessary. It is obvious.

Sufficiency. It follows from Proposition 3.2 that $c_1(S) = c_2(S)$. Then X is an 1-Fréchet space by Corollary 3.26.

Remark 3.28. It follows from Example 3.9 that the collection of n -sequences is a proper subset of the collection of countable subsets of a sequential space (indeed, a metric space). So, Corollary 3.27 and Corollary 3.26, are generations of ([5], Theorem 2) (see Proposition 2.10).

It is well known that the product of a collection of Fréchet spaces is not Fréchet (see Example 2.9). So the following question rises naturally.

Question 3.29. Is the product of a collection of n -Fréchet spaces an n -Fréchet space?

Note that the answer of Question 3.29 is negative with Hausdorff assumption (see [1,2,3 K]).

On the other hand, every Fréchet space and sequential space is precisely the pseudo-open image and the quotient image of a metric space, respectively (see Corollary 2.12 and ([2], Corollary 1.14), and the class of n -Fréchet spaces is between the class of Fréchet spaces the class of sequential spaces (see Remark 3.16). So we have the following question.

Question 3.30. What maps is every n -Fréchet space precisely image of a metric space under?

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Product of powers of quasinormal composition operators

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Abstract Let $T_i, i = 1, 2$ be measurable transformations which define bounded composition operators C_{T_i} on L^2 of a σ -finite measure space. Denote their respective Radon-Nikodym derivatives by $h_i, i = 1, 2$. The main result of the paper is that, if $h_i \circ T_i = h_j$ a.e., for $i, j = 1, 2$, then for each of the positive integers m, n, p the operators $(C_{T_1}^m C_{T_2}^n)^p$ and $(C_{T_2}^m C_{T_1}^n)^p$ are quasinormal. As a consequence, we see that the sufficient condition established in our paper [5] for quasinormality of a composition operator is actually sufficient for all powers to be quasinormal.

Keywords Quasinormal operators, powers of quasinormal composition operators.

§1. Preliminaries

Let (X, Σ, λ) be a sigma-finite measure space and let T be a measurable transformation from X into itself. Set $L^2 = L^2(X, \Sigma, \lambda)$. The equation $C_T f = f \circ T$ for every $f \in L^2$ defines a composition transformation from L^2 to the space of \mathbb{C} -valued functions on X . C_T is a bounded linear operator on L^2 precisely when (i) $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ and (ii) $h = d(\lambda \circ T^{-1})/d\lambda$ is in $L^\infty(X, \Sigma, \lambda) = L^\infty$. Denote by $R(C_T)$ the range of C_T , by C_T^* the adjoint of C_T , and define $T^{-1}(\Sigma)$ as the relative completion of the σ -algebra $\{T^{-1}(A) : A \in \Sigma\}$.

The following lemmas, due to Harrington and Whitley[2], is well known and useful.

Lemma 1.1. Let P denotes the projection of L^2 onto $\overline{R(C_T)}$.

$$(i) \ C_T^* C_T f = h f \text{ and } C_T C_T^* f = (h \circ T) P f, \text{ for all } f \in L^2. \quad (1)$$

$$(ii) \ \overline{R(C_T)} = \{f \in L^2 : f \text{ is } T^{-1}(\Sigma) \text{ measurable}\}. \quad (2)$$

$$(iii) \text{ If } f \text{ is } T^{-1}(\Sigma) \text{ measurable and } g \text{ and } fg \text{ belongs to } L^2, \text{ then } P(fg) = fP(g). \quad (3)$$

Lemma 1.2. With h, T and P as above,

$$\langle (h^n \circ T) P g, g \rangle = \langle (h^n \circ T) g, g \rangle, n \in \mathbb{N}, g \in L^2.$$

Let $B(H)$ denote the Banach algebra of all bounded linear operators on a Hilbert space H . An operator A on a complex Hilbert space H is said to be quasinormal if A commutes with $A^* A$.

Theorem 1.3. Let $C_T, M_\theta \in B(L^2(\lambda))$. Then $C_T M_\theta = M_\theta C_T$ if and only if $\theta = \theta \circ T$ a.e., where M_θ is the multiplication operator induced by θ .

Theorem 1.4. Let $C_{T_1}, C_{T_2} \in B(L^2(\lambda))$ be quasnormal with $h_1 \circ T_2 = h_2$ a.e. Then the products $C_{T_1} C_{T_2}$ and C_{T_2}, C_{T_1} are quasnormal.

§2. Main result and corollaries

In order to prove our main result, it is necessary to state and prove several lemmas. They are stated so that each lemma depends on some subset of the previously stated ones. The proofs which are given contain the essential ideas and may easily be adjusted to give the ones we omit.

Lemma 2.1. If $h \circ T = h$ a.e., then for $n \in N, f \in L^2$, we have

$$\langle h^n f, f \rangle = \langle (C_T^n)^* C_T^n f, f \rangle. \quad (4)$$

Proof. For $n = 1$, the lemma is true by (1). Suppose (4) holds for $n = 1, 2, \dots, k$ and all $f \in L^2$. Then $\langle h^k f, f \rangle = \langle (C_T^k)^* C_T^k f, f \rangle$.

$$\begin{aligned} \text{By induction hypothesis, } \langle h^{k+1} f, f \rangle &= \langle C_T^* C_T (C_T^* C_T)^k f, f \rangle \\ &= \langle h h^k f, f \rangle = \langle (C_T^* C_T)^{k+1} f, f \rangle, \end{aligned}$$

so that (4) holds for $n = k + 1$ and lemma 2.1 follows by induction.

Lemma 2.2. If $h \circ T = h$ a.e., then for all $n \in N, f \in L^2$, we have

$$\langle C_T^n (C_T^n)^* f, f \rangle = \langle (h \circ T)^n f, f \rangle = \langle h^n f, f \rangle.$$

Proof. For $n = 1$, $\langle C_T C_T^* f, f \rangle = \langle (h \circ T) f, f \rangle = \langle h f, f \rangle$.

$$\begin{aligned} \text{Since } \langle C_T^* C_T f, f \rangle &= \langle (h \circ T) P f, f \rangle = \int (h \circ T) |f|^2 dm \\ &= \int h |f|^2 dm = \langle h f, f \rangle. \end{aligned}$$

Since $h \circ T = h$ a.e. By induction hypothesis we can prove lemma 2.2.

Now set $A = C_{T_1}, B = C_{T_2}$, so that the product AB is the operator C_{T_3} .

Lemma 2.3. With A, B, h_1 and h_2 as above, if (a) $h_2 \circ T_2 = h_1$ a.e. (b) $h_1 \circ T_1 = h_1$ a.e., then for each $m, n \in N, f \in L^2$, we have $\langle (A^m B^n)^* (A^m B^n) f, f \rangle = \langle h_2^{m+n} f, f \rangle$.

Proof. First we prove

Claim 2.3.1. With T and h as in §1, for all $r, m \in N, f \in L^2$, we have $\langle (h \circ T)^r C_T^m f, C_T^m f \rangle = \langle h^{r+m} f, f \rangle$. (5)

Proof of claim 2.3.1.

Fix r and induct on m . For $m = 1, f \in L^2$

$$\begin{aligned} \langle (h \circ T)^r C_T f, C_T f \rangle &= \int (h \circ T)^r (|f|^2 \circ T) dm \\ &= \int (h^r |f|^2) h dm = \langle h^{r+1} f, f \rangle. \end{aligned}$$

Suppose (5) holds for $m = k$ and for all $f \in L^2$.

$$\begin{aligned} \text{Then } \langle (h \circ T)^r C_T^{k+1} f, C_T^{k+1} f \rangle &= \langle (h \circ T)^{r+k} C_T f, C_T f \rangle \\ &= \int (h^{r+k} \circ T) (|f|^2 \circ T) dm = \langle h^{r+k+1} f, f \rangle. \end{aligned}$$

So (5) holds for $m = k + 1$. The claim 2.3.1. is proved by induction.

To finish the proof of lemma 2.3, observe that

$$\begin{aligned} \langle (A^m B^n)^* (A^m B^n) f, f \rangle &= \langle (A^m)^* A^m B^n f, B^n f \rangle \\ &= \langle h_1^m B^n f, B^n f \rangle \text{ (by Lemma 2.1)} \end{aligned}$$

$$\begin{aligned}
&= \langle (h_2 \circ T_2)^m B^n f, B^n f \rangle > \text{ (by hypothesis)} \\
&= \langle h_2^{m+n} f, f \rangle > \text{ (by claim 2.3.1)}
\end{aligned}$$

Lemma 2.4. With A, B, h_1 and h_2 as above, if $h_i \circ T_i = h_j, i, j = 1, 2$, then for each $m, n \in N, f \in L^2$, we have $\langle (A^m B^n)(A^m B^n)^* f, f \rangle = \langle h_2^{m+n} f, f \rangle$. (6)

Proof. First we prove

Claim 2.4.1. If $h \circ T = h$ a.e., then for all $r, m \in N, f \in L^2$, we have $\langle h^r (C_T^m)^* f, (C_T^m)^* f \rangle = \langle (h \circ T)^{r+m} f, f \rangle$.

Proof of claim. Fix r and induct on m . For $m = 1$ and $f \in L^2$,

$$\begin{aligned}
\langle h^r C_T^* f, C_T^* f \rangle &= \langle (h \circ T)^r C_T C_T^* f, f \rangle \\
&= \langle (h^{r+1} \circ T) P f, f \rangle > \text{ (by lemma 1.1)} \\
&= \langle ((h^{r+1} \circ T) f, f \rangle > \text{ (by lemma 1.2)}
\end{aligned}$$

By induction hypothesis claim 2.4.1 can be proved.

$$\begin{aligned}
\text{Now } \langle (A^m B^n) (A^m B^n)^* f, f \rangle &= \langle [B^n (B^n)^*] (A^m)^* f, (A^m)^* f \rangle \\
&= \langle (h_2 \circ T_2)^n (A^m)^* f, (A^m)^* f \rangle > \text{ (by lemma 2.2)} \\
&= \langle h_1^n (A^m)^* f, (A^m)^* f \rangle > \text{ (by hypothesis)} \\
&= \langle (h_1 \circ T_1)^{m+n} f, f \rangle > \text{ (by claim 2.4.1)} \\
&= \langle h_2^{m+n} f, f \rangle > \text{ (by hypothesis)}
\end{aligned}$$

Finally, similar technique and the above lemmas may be used to prove the following pair of results, which we collect as Lemma 2.5 and state without proof.

Lemma 2.5. If (6) holds, then for all $m, n, p \in N, f \in L^2$, we have

$$\langle [(A^m B^n)^p]^* [(A^m B^n)^p] f, f \rangle = \langle h_2^{(m+n)p} f, f \rangle \quad (7)$$

and

$$\langle (A^m B^n)^p [(A^m B^n)^p]^* f, f \rangle = \langle h_2^{(m+n)p} f, f \rangle. \quad (8)$$

Remark. $h_2 \circ T_2 = h_2$ is not necessary in the proof of (7). Now we may easily prove our main result.

Lemma 2.6. Let $C_T \in B(L^2(\lambda))$ be quasinormal. Then $M_h^n C_T = C_T M_h^n, n \in N$.

Lemma 2.7. Let C_{T_i}, C_{T_j} are quasinormal with $h_i \circ T_j = h_i$ a.e., $h_j \circ T_i = h_j$ a.e. Then $M_{h_i}^m C_{T_j}^n = C_{T_j}^n M_{h_i}^m$.

In this section, we shall show that the sufficient condition established in the Theorem 2.2[5] for quasinormality of a composition operator is actually sufficient for all powers to be quasinormal.

Theorem 2.8. Let C_{T_i} be quasinormal with $h_i \circ T_j = h_i$ a.e. for $i, j = 1, 2$. Then $(C_{T_1}^m C_{T_2}^n)^p$ and $(C_{T_2}^m C_{T_1}^n)^p$ are also quasinormal for all $m, n, p \in N$.

Proof.

Now

$$\begin{aligned}
(C_{T_1}^m C_{T_2}^n) ((C_{T_1}^m C_{T_2}^n))^* (C_{T_1}^m C_{T_2}^n) &= C_{T_1}^m C_{T_2}^n C_{T_2}^{n*} ((C_{T_1}^m C_{T_1}^m) C_{T_2}^n) \\
&= C_{T_1}^m C_{T_2}^n C_{T_2}^{n*} M_{h_1}^m C_{T_2}^n
\end{aligned}$$

$$\begin{aligned}
&= C_{T_1}^m C_{T_2}^n (C_{T_2}^{n*} C_{T_2}^n) M_{h_1}^m \\
&= C_{T_1}^m C_{T_2}^n M_{h_2}^n M_{h_1}^m \\
&= C_{T_1}^m M_{h_2}^n C_{T_2}^n M_{h_1}^m \\
&= M_{h_2}^n C_{T_1}^m C_{T_2}^n M_{h_1}^m \\
&= C_{T_2}^{n*} C_{T_2}^n C_{T_1}^m C_{T_2}^n M_{h_1}^m \\
&= C_{T_2}^{n*} C_{T_2}^n C_{T_1}^m M_{h_1}^m C_{T_2}^n \\
&= C_{T_2}^{n*} C_{T_2}^n M_{h_1}^m C_{T_1}^m C_{T_2}^n \\
&= C_{T_2}^{n*} M_{h_1}^m C_{T_2}^n C_{T_1}^m C_{T_2}^n \\
&= C_{T_2}^{n*} C_{T_1}^m C_{T_1}^m C_{T_2}^n C_{T_1}^m C_{T_2}^n \\
&= (C_{T_1}^m C_{T_2}^n)^* (C_{T_1}^m C_{T_2}^n) (C_{T_1}^m C_{T_2}^n).
\end{aligned}$$

This implies $C_{T_1}^m C_{T_2}^n$ is a quasinormal composition operator. And similarly $C_{T_2}^n C_{T_1}^m$ is a quasinormal composition operator.

Therefore $(C_{T_1}^m C_{T_2}^n)^p$ and $(C_{T_2}^n C_{T_1}^m)^p$ are also quasinormal composition operators.

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Smarandache magic square

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Abstract This paper contains a magic square. A square array of natural numbers in which the sum of each row and each column is same is a magic square. Smarandache magic square has been defined by Sabin Tabirca [1].

Keywords Magic square.

A Smarandache magic square(SMS) is a square array containing $S(i)$, the Smarandache numbers, only in n rows and m columns such that the sum of each row and each column is same.

The difference between ordinary magic square and SMS is that the elements of SMS are of the form: $S(1), S(2), S(3), \dots, S(n^2)$.

Let (a_{ij}) form the SMS, defined as:

1. $[(a_{ij}), i = 1 \text{ to } n, j = 1 \text{ to } n] = [S(i), i = 1 \text{ to } n^2]$;
2. $\sum a_{ij} = C, j = 1 \text{ to } n$;
3. $\sum a_{ij} = C, i = 1 \text{ to } n$;
4. $\sum S(i) = n \cdot C$, where C is the value of the determinant formed by this SMS.

S. Tabirca has claimed that the SMS exists only for the numbers 6, 7, 9, 58 and 59. The other numbers from $n = 2$ to 100 do not form the SMS. The reason is that the fourth criterion above, i.e. $\sum S(i) = n \cdot C$ is not satisfied.

The following is the table for n and $\sum S(i)$ for which the SMS exists.

| n | $\sum S(i)$ |
|-----|-------------|
| 6 | 330 |
| 7 | 602 |
| 9 | 1413 |
| 58 | 1310162 |
| 69 | 2506080 |

SMS does not exist for $n = 2, 4, 5 \dots$ because for $n = 4$, $\sum S(i)$, for $i = 1$ to 16 is 85, and 4 does not divide 85. Similarly for other values of n .

Here is an example of Smarandache magic square. It is of order 6.

| | | | | | | | Sum of row elements |
|------------------------|----|----|----|----|----|----|---------------------|
| | 3 | 4 | 11 | 11 | 23 | 3 | 55 |
| | 5 | 29 | 5 | 4 | 5 | 7 | 55 |
| | 4 | 13 | 11 | 9 | 5 | 13 | 55 |
| | 5 | 2 | 17 | 6 | 6 | 19 | 55 |
| | 7 | 7 | 7 | 17 | 10 | 7 | 55 |
| | 31 | 0 | 4 | 8 | 6 | 6 | 55 |
| Sum of column elements | 55 | 55 | 55 | 55 | 55 | 55 | 330 |

Now following is the magic square in the form of Smarandache functions.

$$\begin{array}{cccccc}
 S(3) & S(4) & S(11) & S(22) & S(23) & S(6) \\
 S(5) & S(29) & S(10) & S(8) & S(15) & S(7) \\
 S(12) & S(13) & S(33) & S(27) & S(20) & S(26) \\
 S(30) & S(2) & S(17) & S(18) & S(36) & S(19) \\
 S(14) & S(21) & S(35) & S(34) & S(25) & S(28) \\
 S(31) & S(1) & S(24) & S(23) & S(16) & S(9)
 \end{array}$$

Here, $\sum_{i=1}^{36} S(i) = 330$, $n = 6$ and $K = \text{Value of each row/ column of magic square} = 55$, and

$330 = 6 \times 55$. Hence the condition $\sum_{i=1}^{36} S(i) = n.K$ is satisfied.

Therefore the above square is Smarandache magic square.

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Pairwise semi compact and pairwise semi lindeloff spaces

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Abstract In this paper (i, j) semi compact and pairwise semi compact; (i, j) semi Lindeloff and pairwise semi Lindeloff Bitopological spaces are defined and their basic properties are studied.

Keywords Bitopological space, semi open set, (i, j) semi compact, pairwise semi compact, (i, j) semi Lindeloff, pairwise semi Lindeloff Bitopological space.

§1. Introduction

J.C. Kelly[2] introduced the concept of Bitopological spaces in 1963 which paved way to the theory of Bitopological spaces. After him many Authors defined different version of Bitopological spaces and many of their properties like, compactness, connectedness, countability and separation properties were studied with respect to different type of open sets namely semi open, pre open and semi pre (β) open sets. Norman Levine[5] introduced the concept of semi open sets and semi continuity in topological spaces, Maheswari[4] and prasad[4] extended the notions of semi open sets and semi continuity to Bitopological spaces. Shantha Bose[6] further investigated the properties of semi open sets and semi continuity in Bitopological spaces. Ian E. Cooke and Ivan L. Reilly[8] defined and studied the basic properties of compactness in Bitopological spaces. F. H. Khedr et.al.[3] studied interrelations between different open sets between Bitopological spaces. Recently S. Balasubramanian and G. Koteeswara Rao introduced weak and strong Lindeloff Bitopological spaces. In this paper the author introduced compactness using semi open sets in Bitopological spaces which is independent of compactness defined by others and tried to extend the concepts of Lindeloff condition and discussed basic properties in (i, j) and pairwise semi lindeloff spaces.

§2. Preliminaries

A non empty set X together with two topologies τ_1 and τ_2 is called a Bitopological space[2]. Hereafter a space X is called as a Bitopological space unless otherwise stated in this paper. A subset A of X is called (τ_i, τ_j) semi open (briefly (i, j) semi open)[3] if there exists $U \in \tau_i$ such that $U \subset A \subset Cl_j(U)$. A subset A is said to be pairwise semi open if it is (i, j) semi open and (j, i) semi open. A space X is called a pairwise compact[8] if every pairwise open cover

$U \in \tau_i \cup \tau_j$ has a finite sub cover. A space is called a weak (strong) [locally] compact if it is either τ_i or τ_j (τ_i and τ_j) [locally] compact.

Definition 2.1.[6] A subset A of a topological space (X, τ) is said to be a semi open set if there is an open set U of X such that $U \subset A \subset cl(U)$. A function $f: (X, \tau) \rightarrow (Y, \sigma)$ [6] is said to be semi continuous if for each $V \in \sigma$, $f^{-1}(V)$ is semi open in X .

Definition 2.2.[3] $A \subseteq (X, \tau_i, \tau_j)$ is said to be (i,j) semi open if there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$.

Example 1. Let X be the real plane τ_i the half open rectangle topology and τ_j be the usual topology on X . $A = \{(x, y)/0 \leq x < 1, 0 \leq y < 1\} \cup (1, 1)$ is τ_i semi open w.r.to τ_j but neither τ_i open nor τ_j open.

Example 2. Let X be the real line τ_i be the lower limit topology on X and τ_j the usual topology on X . $A = \{x/0 \leq x < 1\}$ is τ_i semi open w.r.to τ_j but neither τ_i open nor τ_j open.

Definition 2.3. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a function then f is said to be pairwise continuous[resp. pairwise open map] if the induced functions $f: (X, \tau_i) \rightarrow (Y, \sigma_i)$ and $f: (X, \tau_j) \rightarrow (Y, \sigma_j)$ are both continuous [resp. open].

Definition 2.4.[3] Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a single valued function then f is said to be pairwise semi continuous if the inverse image of each σ_i -open set of Y is (i, j) semi open in X , where $i \neq j$, $i, j = 1, 2$.

Theorem 2.5.[3] Let A be a subset of (X, τ_i, τ_j) . If there exists an (i, j) semi open set U such that $U \subset A \subset cl_j(U)$, then A is (i, j) semi open.

Theorem 2.6.[3] The union of arbitrarily many (i, j) semi open sets is (i, j) semi open.

Proof. Let $\{A_k/k \in K\}$ be a collection of (i, j) semi open sets in (X, τ_i, τ_j) . For each $k \in K$, \exists $a\tau_i$ -open set U_k such that $U_k \subset A_k \subset cl_j(U_k)$, then $\cup U_k \subset \cup A_k \subset \cup cl_j(U_k) = cl_j(\cup U_k)$. Put $U = \cup U_k$, U is τ_i -open. Thus $\cup A_k$ is (i, j) semi open.

Theorem 2.7.[3] Let $A \subset Y \subset X$ where X is a Bitopological space. If A is (i, j) semi open in X , it is (i, j) semi open in Y .

Theorem 2.8.[3] Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be pairwise continuous pairwise open map. If A is (i, j) semi open in X then $f(A)$ is (i, j) semi open in Y .

Proof. Let A be (i, j) semi open in X , then there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$. Now $f(U) \subset f(A) \subset f(cl_j(U))$, $f(U)$ is σ_i -open, f being open map and $f(cl_j(U)) \subset cl_j(f(U))$, f being continuous. $f(A)$ is (i, j) semi open in X .

Theorem 2.9.[3] Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be pairwise semi continuous pairwise open map. If A is (i, j) semi open in Y then $f^{-1}(A)$ is (i, j) semi open in X .

Proof. Let A be (i, j) semi open in Y , then there exists $W \in \sigma_i$ such that $W \subset A \subset cl_j(W)$. Since f is pairwise open, it follows that $f^{-1}(W) \subset f^{-1}(A) \subset f^{-1}(cl_j(W)) \subset cl_j(f^{-1}(W))$. Since f is pairwise semi continuous, $f^{-1}(W)$ is (i, j) semi open in X . By theorem 2.5, $f^{-1}(A)$ is (i, j) semi open in X .

Remark 1. [3] τ_i open \Rightarrow (i, j) semi open.

§3. (i, j) Semi compact Bitopological space

Definition 3.1. (X, τ_i, τ_j) is said to be (i) (i, j) semi compact if each (i, j) semi open cover has a finite (i, j) semi open sub cover. (ii) locally (i, j) semi compact if each $x \in X$ has (i, j) semi-open neighborhood with compact (i, j) semi closure.

Example 3. since every open set is (i, j) semi open, example $X = R, \tau_i = R_\omega, \tau_j = R_\omega$; where R_ω = minimum well ordered uncountable set will act as the example for these two definitions.

Example 4. Let $X = [a, b]$ be the subspace of real numbers with $\tau_i = \{[a, c)/a < c < b\}$ and $\tau_j = \{(c, d)/c < d\}$ as basic open sets then, (X, τ_i, τ_j) is (i, j) semi compact Bitopological space.

Theorem 3.2. Every (i, j) semi compact space is locally (i, j) semi compact.

Theorem 3.3. Let (X, τ_i, τ_j) be Bitopological space and $A \subset (X, \tau_i, \tau_j)$. Then A is (i, j) semi compact subset of (X, τ_i, τ_j) if the subspace $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi compact.

Theorem 3.4. A τ_i -closed subspace of a [locally] (i, j) semi compact space is [locally] (i, j) semi compact. The product of two [locally] (i, j) semi compact spaces is [locally] (i, j) semi compact.

Proof. $A \subseteq (X, \tau_i, \tau_j)$ is τ_i -closed and (X, τ_i, τ_j) is (i, j) semi compact. For $U = \{U_i/i \in I\}$ is (i, j) semi open cover of A we can find $U' = \{G(U)/U \in U\} \cup \{X - A\}$ an (i, j) semi open cover of X , where G is such that $U = G \cap A$, which implies U' admits finite sub cover giving rise to the finite sub covers for U . Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi compact.

$A \subseteq (X, \tau_i, \tau_j)$ is τ_i -closed, (X, τ_i, τ_j) is locally (i, j) semi compact and $x \in A \subseteq X$ be any point. Then x has (i, j) semi compact neighborhood G . Since G is (i, j) semi compact, by theorem 3.3 we can find (i, j) semi compact neighborhood $U = G \cap A$. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is locally (i, j) semi compact.

Direct consequence of Tychonoff's theorem.

Theorem 3.5. Let (X, τ_i, τ_j) be any [locally] (i, j) semi compact Bitopological space. Then (X, τ_i) is [locally] compact.

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X . since every τ_i open set is (i, j) semi open set, τ_i open cover will become (i, j) semi open cover and this contains finite (i, j) semi open cover. Since (X, τ_i, τ_j) is (i, j) semi compact. Therefore (X, τ_i) is compact. Similar argument gives the proof of the second part.

Corollary 3.6. If (X, τ_i, τ_j) be any (i, j) semi compact Bitopological space. Then (X, τ_i) is locally compact.

Theorem 3.7. A Bitopological space is (i, j) semi compact iff (i) Every class of (i, j) semi closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of (i, j) semi closed sets with finite intersection property has non empty intersection. (iii) iff Every basic (i, j) semi open cover has a finite sub cover.

Corollary 3.8. A Bitopological space is (i, j) semi compact iff (i) Every class of τ_i -closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of τ_i -closed sets with finite intersection property has non empty intersection. (iii) iff Every τ_i -basic open cover has a finite sub cover.

Theorem 3.9. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (X, τ_i, τ_j) is (i, j) semi compact. Then $f(X)$ is (i, j) semi compact.

Proof. Let $\{G_i/i \in I\}$ be (i, j) semi open cover for $f(X)$. Since f is pairwise semi continuous, $\{f^{-1}(G_i)/i \in I\}$ is (i, j) semi open cover for X and has a finite (i, j) semi open sub cover. Therefore, for f is pairwise open, $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f(X)$ is (i, j) semi compact.

Theorem 3.10. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (Y, σ_i, σ_j) is (i, j) semi compact. Then $f^{-1}(Y)$ is (i, j) semi compact.

Proof. Let $\{G_i/i \in I\}$ be (i, j) semi open cover for $f^{-1}(Y)$ implies each G_i is (i, j) semi open set in $f^{-1}(Y)$, then there exists $U \in \tau_i$ such that $U \subset A \subset cl_j(U)$. Since f is pairwise semi continuous, we have $f(U) \subset f(A) \subset f(cl_j(U)) \subset cl_j(f(U))$. This shows that $f(G_i)$ is (i, j) semi open in Y and hence $\{f(G_i)/i \in I\}$ is (i, j) semi open cover for Y and has a finite (i, j) semi open sub cover. Since (Y, σ_i, σ_j) is (i, j) semi compact. Therefore $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f^{-1}(Y)$ is (i, j) semi compact.

Corollary 3.11. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be pairwise continuous pairwise open map and (Y, σ_i, σ_j) is (i, j) semi compact. Then $f^{-1}(Y)$ is (i, j) semi compact.

Proof. since every pairwise continuous map is pairwise semi continuous, result follows directly from the above theorem.

Remark 2. [locally] (i, j) semi compact is weak hereditary.

§4. Pairwise semi compact Bitopological space

Definition 4.1. (X, τ_i, τ_j) is said to be (i) pairwise semi compact if it is (i, j) and (j, i) semi compact. (ii) locally pairwise semi compact if it is locally (i, j) semi compact and locally (j, i) semi compact.

Example 5. Let R be the space of reals with topologies τ_i and τ_j , where τ_i has $(a, b]$ as basic open sets, and τ_j has $[a, b)$, $a < b$ as basic open sets, then (X, τ_i, τ_j) pairwise semi compact.

Example 6. since every open set is (i, j) semi open, (X, τ_i, τ_j) , where $X = R, \tau_i = \{G_n/G_n = (-n, n), n \in \mathbb{Z}^+\}; \tau_j = R_\Omega$ will act as the example for these two definitions.

Theorem 4.2. Every pairwise semi compact space is locally pairwise semi compact.

Proof. obvious from the definitions of pairwise semi compact and locally pairwise semi compact spaces.

Theorem 4.3. Let (X, τ_i, τ_j) be Bitopological space and $A \subseteq (X, \tau_i, \tau_j)$. Then A is pairwise semi compact subset of (X, τ_i, τ_j) iff the subspace $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi compact

Theorem 4.4. A Bi-closed subspace of a [locally] pairwise semi compact space is [locally] pairwise semi compact. The product of two [locally] pairwise semi compact spaces is [locally] pairwise semi compact.

Proof. $A \subseteq (X, \tau_i, \tau_j)$ is Bi-closed and (X, τ_i, τ_j) is pairwise semi compact. Let $U = \{U_i/i \in I\}, V = \{V_i/i \in I\}$ are respectively (i, j) and (j, i) semi open covers of A with respect to $\tau_{i/A}$ and $\tau_{j/A}$ then for each $U_i \in U$ and for each $V_i \in V$, fix an (i, j) semi open set G and an (j, i) semi open set H such that $U = G \cap A, V = H \cap A$, then the families $U' = \{G(U)/U \in$

$U\} \cup \{X - A\}$ and $V' = \{H(V)/V \in V\} \cup \{X - A\}$ are (i,j) and (j, i) semi open covers of X with respect to τ_i and τ_j respectively, which implies U' and V' admits finite sub covers which in turn gives rise to the finite sub covers for U and V respectively. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi compact

Direct from Tychnoff's theorem.

Theorem 4.5. Let (X, τ_i, τ_j) be any [locally] pairwise semi compact Bitopological space. Then (X, τ_i) and (X, τ_j) are [locally] compact.

Corollary 4.6. If (X, τ_i, τ_j) be any pairwise semi compact Bitopological space. Then (X, τ_i) and (X, τ_j) are locally compact.

Theorem 4.7. A Bitopological space is Pairwise semi compact iff (i) Every class of pairwise closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of pairwise closed sets with finite intersection property has non empty intersection. (iii) iff Every pairwise basic open cover has a finite sub cover.

Theorem 4.8. A Bitopological space is Pairwise semi compact iff (i) Every class of Bi-closed sets with empty intersection has a finite subclass with empty intersection. (ii) iff Every class of Bi-closed sets with finite intersection property has non empty intersection. (iii) iff Every pairwise basic open cover has a finite sub cover.

Theorem 4.9. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (X, τ_i, τ_j) is pairwise semi compact. Then $f(X)$ is pairwise semi compact.

Proof. Let $\{G_i/i \in I\}$ be pairwise semi open cover for $f(X)$. Then $\{G_i/i \in I\}$ is both (i,j) and (j,i) semi open covers for $f(X)$. since f is pairwise semi continuous, $\{f^{-1}(G_i)/i \in I\}$ is pairwise semi open cover for X and has a finite pairwise semi open sub cover. Therefore, for f is pairwise open, $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f(X)$ is pairwise semi compact.

Theorem 4.10. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be a pairwise semi continuous pairwise open map and (Y, σ_i, σ_j) is pairwise semi compact. Then $f^{-1}(Y)$ is pairwise semi compact.

Proof. Let $\{G_i/i \in I\}$ be pairwise semi open cover for $f^{-1}(Y)$ implies $\{f(G_i)/i \in I\}$ is pairwise semi open cover for Y and has a finite pairwise semi open sub cover. Since (Y, σ_i, σ_j) is pairwise semi compact. Therefore $\{G_i/i \in I\}$ has a finite semi open sub cover. Thus $f^{-1}(Y)$ is pairwise semi compact.

Theorem 4.11. Let $f: (X, \tau_i, \tau_j) \rightarrow (Y, \sigma_i, \sigma_j)$ be pairwise continuous pairwise open map and (Y, σ_i, σ_j) is pairwise semi compact. Then $f^{-1}(Y)$ is pairwise semi compact.

Proof. since every pairwise continuous map is pairwise semi continuous, result follows directly from the above theorem.

Example 7. $X = R$; $\tau_i = \{\phi, R, (a, \infty) : a \in R\}$; $\tau_j = \{\phi, R, (-\infty, a) : a \in R\}$ then (X, τ_i, τ_j) is not (i,j) and pairwise semi compact. But according to Cooke and Reilly it is pairwise compact.

Example 8. $X = [0, 1]$; $\tau_i = \{\phi, X, \{0\}, [0, a) : a \in X\}$; $\tau_j = \{\phi, X, \{1\}, (a, 1] : a \in X\}$ then (X, τ_i, τ_j) is pairwise semi compact. According to Cooke and Reilly it is pairwise B-compact.

Example 9. $X = \{a, b, c\}$; $\tau_i = \{\phi, X, \{a, b\}, \{c\}\}$; $\tau_j = \{\phi, X, \{a\}, \{b, c\}\}$ then (X, τ_i, τ_j) is not B-compact according to Cooke and Reilly. Since $\{\{a\}, \{b, c\}\}$ is a τ_j open

cover of X without τ_i open sub cover.

Example 10. $X = R^2$; τ_i = usual topology; τ_j = Half open rectangle topology; $A = \{(x, y)/0 \leq x < 1; 0 \leq y < 1\} \cup (1, 1)$ is τ_j semi open w.r.to τ_i but neither τ_i open nor τ_j open.

Remark 3. [locally] pairwise semi compact is weak hereditary.

§5. (i, j) Semi second countability

Definition 5.1.

- (i) (X, τ_i, τ_j) is said to be (i, j) semi second countable if it has countable (i, j) semi open base.
- (ii) (X, τ_i, τ_j) is said to be (i, j) semi first countable if each x in X has countable (i, j) semi open base.

Example 11. (X, τ_i, τ_j) is (i, j) semi second countable, where $X = R$, $\tau_i = R_i$; $\tau_j = \{G_n/G_n = (-n, n) : n \in N\}$.

Theorem 5.2.

- (i) Any subspace of (i, j) semi [first] second countable space is (i, j) semi [first] second countable.
- (ii) Any countable product of (i, j) semi [first] second countable spaces is (i, j) semi [first] second countable.

Proof. (i) Let $(A, \tau_{i/A}, \tau_{j/A})$ be any subspace of an (i, j) semi second countable space (X, τ_i, τ_j) and since (X, τ_i, τ_j) is (i, j) semi second countable, there exists countable (i, j) semi-open base $\{G_i\}$, which in turn gives $H_i = G_i \cap A$ a countable $(i/A, j/A)$ semi open base for $(A, \tau_{i/A}, \tau_{j/A})$. Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi second countable. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi first countable.

(ii) Let $X = \Pi X_i$; X'_i 's are (i, j) semi second countable spaces and $B_i = \{B_i^j/j \in J\}$ is countable (i, j) semi open base for each X_i with respect to τ_1^i and τ_2^i respectively. Then $B = \Pi B_i = \{U_i/U_i = \Pi B_i^j \text{ where } B_i^j \in B_i \text{ for finite } i \text{ and } B_j = X_j \text{ for } j \neq i\}$, is the countable (i, j) semi open base in (X, τ_1, τ_2) where $\tau_1 = \Pi_{i \in I} \tau_1^i$, $\tau_2 = \Pi_{i \in I} \tau_2^i$. Hence $X = \Pi X_i$ is (i, j) semi second countable. Similar argument shows $X = \Pi X_i$ is (i, j) semi first countable.

Corollary 5.3. Let (X, τ_i, τ_j) be any (i, j) semi [first] second countable Bitopological space. (i) Any subspace of (X, τ_i) [first] second countable space is (i, j) semi [first] second countable. (ii) Any countable product of (X, τ_i) [first] second countable spaces is (i, j) semi [first] second countable.

Remark 4. The property of (i, j) semi [first] second countability is hereditary.

§6. Pairwise semi second countability

Definition 6.1. (X, τ_i, τ_j) is said to be pairwise semi second [first] countable if it is both (i, j) and (j, i) semi second [first] countable.

Example 12. X = countable set; τ_i = cofinite topology and τ_j = discrete topology, is pairwise semi second countable.

Theorem 6.2. (i) Any subspace of pairwise semi [first] second countable space is pairwise semi [first] second countable. (ii) Any countable product of pairwise semi [first] second countable spaces is pairwise semi [first] second countable.

Proof. (i) Let $(A, \tau_{i/A}, \tau_{j/A})$ be any subspace of pairwise semi second countable space (X, τ_i, τ_j) . $B = \{B_i/i \in I\}$, $B' = \{B'_i/i \in I\}$ are respectively countable (i, j) and (j, i) semi open bases in X . Then $C = \{C_i/C_i = B_i \cap A\}_{i \in I}$, $C' = \{C'_i/C'_i = B'_i \cap A\}_{i \in I}$ are countable $(i/A, j/A)$ and $(j/A, i/A)$ semi open bases for A . Therefore $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi second countable. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi first countable.

(ii) Let $X = \Pi X_i$; X_i 's are pairwise semi second countable, $B_i = \{B_i^j/j \in J\}$ and $B'_i = \{B'^j_{ij}/j \in J\}$ are countable (i, j) and (j, i) semi open bases for each X_i with respect to τ_1^i and τ_2^i respectively. Then $B = \Pi B_i = \{U_i/U_i = \Pi B_{ij} \text{ where } B_i^j \in B_i \text{ for finite } i, \text{ and } B_j = X_j \text{ for } j \neq i\}$, $B' = \Pi B'_i = \{U'_i/U'_i = \Pi B'_{ij} \text{ where } B'^j_{ij} \in B'_i \text{ for finite } i, \text{ and } B'_j = X_j \text{ for } j \neq i\}$ are respectively the countable (i, j) and (j, i) semi open bases in (X, τ_1, τ_2) where $\tau_1 = \Pi_{i \in I} \tau_1^i$, $\tau_2 = \Pi_{i \in I} \tau_2^i$. Hence $X = \Pi X_i$ is pairwise semi second countable. Similar argument shows $X = \Pi X_i$ is pairwise semi first countable.

Proofs of the following are trivial and so left to the reader.

Definition 6.3. (X, τ_i, τ_j) is said to be Bi second countable if it is both τ_i and τ_j second countable.

Corollary 6.4. Let (X, τ_i, τ_j) be any Bitopological space. (i) Every Bi [first] second countable Bitopological space is pairwise semi [first] second countable.

Corollary 6.5. Let (X, τ_i, τ_j) be any Bitopological space. (i) Any subspace of Bi first [second] countable space is pairwise semi first [second] countable. (ii) Any countable product of Bi first [second] countable space is pairwise semi first [second] countable.

Remark 5. The property of pairwise semi [first] second countability is hereditary.

§7. (i, j) semi Lindeloff Bitopological space

Definition 7.1. (X, τ_i, τ_j) is said to be (i, j) semi Lindeloff if each (i, j) semi open cover has a countable (i, j) semi open sub cover and (X, τ_i, τ_j) is locally (i, j) semi Lindeloff if each $x \in X$ has (i, j) semi Lindeloff neighborhood.

Example 13. Let X be the real line τ_i the lower limit topology and τ_j be the upper limit topology on X , then (X, τ_i, τ_j) is (i, j) semi Lindeloff.

Theorem 7.2. Suppose (X, τ_i, τ_j) has a countable (i, j) semi open base then every semi open cover of (X, τ_i, τ_j) contains a countable sub collection covering (X, τ_i, τ_j) .

Proof. Let $B = \{B_n/n \in N\}$ is (i, j) countable semi open base for X . Let $U = \{U_i/i \in I\}$ be (i, j) semi open covering of X . For each $n \in N$, choose an element U_n of U such that $B_n \subseteq U_n$. Then the collection $U' = \{U_n/n \in N\}$ is countable. Furthermore, it covers X . Given a point $x \in X$, we can choose an element U_n of U containing x . Since U_n is (i, j) semi open, there exists basis elements B_n , such that $x \in B_n \subseteq U_n$. Since $B_n \subseteq U_n$, it contains x . Therefore $x \in B_n \subseteq U_n$. Thus U' is a countable sub collection of U which covers X with respect to τ_i and τ_j respectively. Hence the theorem.

Theorem 7.3. (i) A τ_i -closed subspace of a [locally] (i, j) semi Lindeloff space is [locally] (i, j) semi Lindeloff. (ii) The product of two [locally] (i, j) semi Lindeloff spaces need not be [locally] (i, j) semi Lindeloff.

Proof. (i) Let A be τ_i -closed subspace of (i, j) semi Lindeloff space (X, τ_i, τ_j) . Let $U =$

$\{U_i/i \in I\}$ be $(i/A, j/A)$ semi open cover of A then for each $U_i \in U$ fix an (i, j) semi open set G such that $U = G \cap A$, then the family $U' = \{G_i/i \in I\} \cup \{X - A\}$ is an (i, j) semi open cover of X , which implies U' admits countable sub cover which in turn gives countable $(i/A, j/A)$ sub cover for A . Hence $(A, \tau_{i/A}, \tau_{j/A})$ is (i, j) semi Lindeloff. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is locally (i, j) semi Lindeloff.

(ii) Remark 1 and standard theorem (sorgenfrey plane) gives the proof for this part.

Theorem 7.4. Every (i, j) semi [first] second countable space is [locally] (i, j) semi Lindeloff.

Corollary 7.5. Let (X, τ_i, τ_j) be any Bitopological space. If (X, τ_i) is [first] second countable then (X, τ_i, τ_j) is [locally] (i, j) semi Lindeloff.

Note. the converse of the above theorem is not true.

Theorem 7.6. If (X, τ_i, τ_j) is [locally] (i, j) semi compact Lindeloff Bitopological space, then (X, τ_i) is [locally] Lindeloff topological space.

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i, j) semi open cover for X , then for (X, τ_i, τ_j) is (i, j) semi Lindeloff, $\{G_i/i \in I\}$ will have a countable sub cover which in turn becomes countable open cover for (X, τ_i) . Hence (X, τ_i) is Lindeloff.

Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i, j) semi open cover for X , then for (X, τ_i, τ_j) is (i, j) semi compact, $\{G_i/i \in I\}$ will have a finite sub cover which in turn becomes finite open cover for (X, τ_i) . Hence (X, τ_i) is Lindeloff. (since every compact space is Lindeloff).

Similar argument gives the proof of locally Lindeloffness.

Corollary 7.7. If (X, τ_i, τ_j) is (i, j) semi compact Lindeloff Bitopological space, then (X, τ_i) is locally Lindeloff topological space.

Theorem 7.8. If (X, τ_i, τ_j) is [locally] (i, j) semi Lindeloff and (Y, σ_i, σ_j) is [locally] (i, j) semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is [locally] (i, j) semi Lindeloff.

Theorem 7.9. Every [locally] (i, j) semi compact space is [locally] (i, j) semi Lindeloff.

Remark 6. [locally] (i, j) semi Lindeloff is weak hereditary.

§8. Pairwise semi Lindeloff Bitopological space

Definition 8.1. (i) (X, τ_i, τ_j) is said to be pairwise semi Lindeloff if it is (i, j) and (j, i) semi Lindeloff and (ii) (X, τ_i, τ_j) is said to be locally pairwise semi Lindeloff if it is locally (i, j) semi Lindeloff and locally (j, i) semi Lindeloff.

Example 14. Let X be the real line τ_i the lower limit topology and τ_j be the upper limit topology on X , then (X, τ_i, τ_j) is pairwise semi Lindeloff.

Example 15. Let $X = R$; $\tau_i = R_\ell$ the lower limit topology and $\tau_j = R_\Omega$ where $R_\Omega =$ minimum well ordered uncountable set topology on X , then (X, τ_i, τ_j) is pairwise semi Lindeloff.

Theorem 8.2. Suppose (X, τ_i, τ_j) has a countable pairwise semi open base then every semi open cover of (X, τ_i, τ_j) contains a countable sub collection covering (X, τ_i, τ_j) .

Proof. Let $B = \{B_n/n \in N\}$, $B' = \{B'_n/n \in N\}$ are (i, j) and (j, i) countable semi open bases for X respectively. Let $U = \{U_i/i \in I\}$, $V = \{V_i/i \in I\}$ be (i, j) and (j, i) semi open coverings of X . For each $n \in N$, choose an element U_n of U ; V_n of V such that $B_n \subseteq U_n$ and

$B'_n \subseteq V_n$ respectively. Then the collection $U' = \{U_n/n \in N\}$; $V' = \{V_n/n \in N\}$ are countable. Furthermore, they covers X with respect to (i, j) and (j, i) respectively. Given a point $x \in X$, we can choose an element U_n of U containing x and V_n of V containing x . Since U_n, V_n are (i, j) and (j, i) semi open respectively, there exists basis elements B_n, B'_n such that $x \in B_n \subseteq A$ and $x \in B'_n \subseteq A$. Since $B_n \subseteq U_n, B'_n \subseteq V_n$ they both contains x . Therefore $x \in B_n \subseteq U_n$ and $x \in B'_n \subseteq V_n$. Thus U' is a countable sub collection of U ; V' is a countable sub collection of V , which covers X with respect to τ_1 and τ_2 respectively. Hence the theorem.

Definition 8.3. A subset in (X, τ_i, τ_j) is said to be a Bi-closed set if it is both τ_i and τ_j closed.

Theorem 8.4. (i) A Bi-closed subspace of a [locally] pairwise semi Lindeloff space is [locally] pairwise semi Lindeloff. (ii) The product of two [locally] pairwise semi Lindeloff spaces need not be [locally] pairwise semi Lindeloff.

Proof. (i) Let A be Bi-closed subspace of pairwise semi Lindeloff space (X, τ_i, τ_j) . Let $U = \{U_i/i \in I\}, V = \{V_i/i \in I\}$ are respectively $(i/A, j/A)$ and $(j/A, i/A)$ semi open covers of A then for each $U_i \in U$ and for each $V_i \in V$, fix an (i, j) semi open set G and (j, i) semi open set H such that $U = G \cap A, V = H \cap A$, then the families $U' = \{G_i/i \in I\} \cup \{X - A\}$ and $V' = \{H_i/i \in I\} \cup \{X - A\}$ are (i, j) and (j, i) semi open covers of X respectively, which implies U' and V' admits countable sub covers which in turn gives countable sub covers for A . Hence $(A, \tau_{i/A}, \tau_{j/A})$ is pairwise semi Lindeloff. Similar argument shows $(A, \tau_{i/A}, \tau_{j/A})$ is locally pairwise semi Lindeloff.

(ii) Let (X, τ_i, τ_j) be pairwise semi Lindeloff space, where $X = R, \tau_i = R_u, \tau_j = R_v$, then $X \times X$ is the product of two pairwise semi Lindeloff spaces called Sorgenfrey planes. It has basis all sets of the form $[a, b)X[c, d)$ and $(b, -a]X(d, c]$ in the planes. Consider the subspaces $L = \{xX(-x)/x \in R_u\}$ and $L' = \{(-x)Xx/x \in R_v\}$ in R_u^2 and R_v^2 respectively. Then obviously L and L' are (i, j) and (j, i) semi closed in R_u^2 and R_v^2 respectively. By covering R^2 by the (i, j) semi open sets $R_u^2 - L$ and by the basis elements of the form $[a, b)X[-a, d)$ and by (j, i) semi open sets $R_v^2 - L'$ and by the basis elements of the form $(b, -a]X[d, a)$ we can see that each of these basis elements in R_u^2 and R_v^2 intersects L and L' in at most one point respectively. Since both L and L' are uncountable, no countable sub collection covers R_u^2 and R_v^2 respectively. Thus the product of two pairwise semi Lindeloff Bitopological spaces is not pairwise semi Lindeloff.

Theorem 8.5. Every Bi - [first] second countable space is [locally] pairwise semi Lindeloff.

Theorem 8.6. If (X, τ_i, τ_j) is [locally] pairwise semi compact Lindeloff Bitopological space, then (X, τ_i) and (X, τ_j) are [locally] Lindeloff topological spaces

Proof. Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i, j) semi open cover for X , then for (X, τ_i, τ_j) is (i, j) semi Lindeloff, $\{G_i/i \in I\}$ will have a countable sub cover which in turn becomes countable open cover for (X, τ_i) . Hence (X, τ_i) is Lindeloff.

Let $\{G_i/i \in I\}$ be any τ_i open cover for X implies $\{G_i/i \in I\}$ is (i, j) semi open cover for X , then for (X, τ_i, τ_j) is (i, j) semi compact, $\{G_i/i \in I\}$ will have a finite sub cover which in turn becomes finite open cover for (X, τ_i) . Hence (X, τ_i) is Lindeloff (since every compact space is Lindeloff).

Similar argument gives the proof of locally Lindeloffness.

Corollary 8.7. If (X, τ_i, τ_j) is pairwise semi (compact)Lindeloff Bitopological space, then

(X, τ_i) and (X, τ_j) are locally Lindeloff topological space.

Theorem 8.8. If (X, τ_i, τ_j) is [locally] pairwise semi Lindeloff and (Y, σ_i, σ_j) is [locally] pairwise semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is [locally] pairwise semi Lindeloff.

Corollary 8.9. If (X, τ_i, τ_j) is pairwise semi Lindeloff and (Y, σ_i, σ_j) is pairwise semi compact then $(X \times Y, \tau_i \times \sigma_i, \tau_j \times \sigma_j)$ is pairwise locally semi Lindeloff.

Theorem 8.10. Every [locally] pairwise semi compact space is [locally] pairwise semi Lindeloff.

Proof. Let (X, τ_i, τ_j) is pairwise semi compact. Then for G_i an (i, j) and G'_i an (j, i) semi open covers with finite (i, j) and (j, i) semi open covers for X respectively. Since every finite cover is countable cover, G_i an (i, j) and G'_i an (j, i) are semi open covers with countable (i, j) and (j, i) semi open sub covers for X respectively. Hence (X, τ_i, τ_j) is pairwise semi Lindeloff. Trivial verification shows (X, τ_i, τ_j) is [locally] pairwise semi Lindeloff.

Remark 7. [locally] pairwise semi Lindeloff is weak hereditary.

Conclusion. In this paper we defined new compactness and Lindeloffness in Bitopological spaces and studied their interrelations.

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A note on f -minimum functions

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Abstract For a given arithmetical function $f : \mathbb{N} \rightarrow \mathbb{N}$, let $F : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $F(n) = \min\{m \geq 1 : n|f(m)\}$, if this exists. Such functions, introduced in [4], will be called as the f -minimum functions. If f satisfies the property $a \leq b \implies f(a)|f(b)$, we shall prove that $F(ab) = \max\{F(a), F(b)\}$ for $(a, b) = 1$. For a more restrictive class of functions, we will determine $F(n)$ where n is an even perfect number. These results are generalizations of theorems from [10], [1], [3], [6].

Keywords Divisibility of integers, prime factorization, arithmetical functions, perfect numbers.

§1. Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and $f : \mathbb{N} \rightarrow \mathbb{N}$ a given arithmetical function, such that for each $n \in \mathbb{N}$ there exists at least an $m \in \mathbb{N}$ such that $n|f(m)$. In 1999 and 2000 [4], [5], as a common generalization of many arithmetical functions, we have defined the application $F : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$F(n) = \min\{m \geq 1 : n|f(m)\}, \quad (1)$$

called as the " f -minimum function". Particularly, for $f(m) = m!$ one obtains the Smarandache function (see [10], [1])

$$S(n) = \min\{m \geq 1 : n|m!\}. \quad (2)$$

Moree and Roskam [2], and independently the author [4], [5], have considered the Euler minimum function

$$E(n) = \min\{m \geq 1 : n|\varphi(m)\}, \quad (3)$$

where φ is Euler's totient. Many other particular cases of (1), as well as, their "dual" or analogues functions have been studied in the literature; for a survey of concepts and results, see [9].

In 1980 Smarandache discovered the following basic property of $S(n)$ given by (2):

$$S(ab) = \max\{S(a), S(b)\} \text{ for } (a, b) = 1. \quad (4)$$

Our aim in what follows is to extend property (4) to a general class of f -minimum functions. Further, for a subclass we will be able to determine $F(n)$ for even perfect numbers n .

§2. Main results

Theorem 1. Suppose that F of (1) is well defined. Then for distinct primes p_i , and arbitrary $\alpha_i \geq 1$ ($i = 1, 2, \dots, r$) one has

$$F\left(\prod_{i=1}^r p_i^{\alpha_i}\right) \geq \max\{F(p_i^{\alpha_i}) : i = 1, 2, \dots, r\}. \quad (5)$$

The second result offers a reverse inequality:

Theorem 2. With the notations of Theorem 1 suppose that f satisfies the following divisibility condition:

$$a|b \implies f(a)|f(b) \quad (a, b \geq 1) \quad (*)$$

Then one has

$$F\left(\prod_{i=1}^r p_i^{\alpha_i}\right) \leq l.c.m.\{F(p_i^{\alpha_i}) : i = 1, 2, \dots, r\}, \quad (6)$$

where $l.c.m.$ denotes the least common multiple.

By replacing $(*)$ with another condition, a more precise result is obtainable:

Theorem 3. Suppose that f satisfies the condition:

$$a \leq b \implies f(a)|f(b) \quad (a, b \geq 1). \quad (**)$$

Then

$$F(mn) = \max\{F(m), F(n)\} \text{ for } (m, n) = 1. \quad (7)$$

Finally, we shall prove the following:

Theorem 4. Suppose that f satisfies $(**)$ and the following two assumptions:

$$(i) \ n|f(n); \ (ii) \ \text{For each prime } p \text{ and } m < p \text{ we have } p \nmid f(n). \quad (8)$$

Let k be an even perfect number. Then

$$F(k) = k/2^s, \text{ where } 2^s || k. \quad (9)$$

Remarks . (1) The function φ satisfies property $(*)$. Then relation (6) gives a result for the Euler minimum function $E(n)$ (see [7], [8]).

(2) Let $f(m) = m!$. Then clearly $(**)$ holds true. Thus (7) extends relation (4). For another example, let $f(m) = l.c.m.\{1, 2, \dots, m\}$. Then the function F given by (1) satisfies again (7), proved e.g. in [1].

(3) If $f(n) = n!$, then both (i) and (ii) of (8) are satisfied. This relation (9) for $F \equiv S$ follows. This was first proved in [3] (see also [6]).

§3. Proof of theorems

Theorem 1. There is no loss of generality to prove (5) for $r = 2$. Let p^α, q^β be two distinct prime powers. Then

$$F(p^\alpha q^\beta) = \min\{n \geq 1 : p^\alpha q^\beta | f(n)\} = m_0,$$

so $p^\alpha q^\beta | f(m_0)$. This is equivalent to $p^\alpha | f(m_0)$, $q^\beta | f(m_0)$. By definition (1) we get $m_0 \geq F(p^\alpha)$ and $m_0 \geq F(q^\beta)$, i.e. $F(p^\alpha q^\beta) \geq \max\{F(p^\alpha), F(q^\beta)\}$. It is immediate that the same proof applies to $F\left(\prod p^\alpha\right) \geq \max\{F(p^\alpha)\}$, where p^α are distinct prime powers.

Theorem 2. Let $F(p^\alpha) = m_1$, $F(q^\beta) = m_2$. By definition (1) of function F it follows that $p^\alpha | F(m_1)$ and $q^\beta | F(m_2)$. Let $\text{l.c.m.}\{m_1, m_2\} = g$. Since $m_1 | g$, one has $f(m_1) | f(g)$ by (*). Similarly, since $m_2 | g$, one can write $f(m_2) | f(g)$. These imply $p^\alpha | f(m_1) | f(g)$ and $q^\beta | f(m_2) | f(g)$, yielding $p^\alpha q^\beta | f(g)$. By definition (1) this gives $g \geq F(p^\alpha q^\beta)$, i.e. $\text{l.c.m.}\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$, proving the theorem for $r = 2$. The general case follows exactly by the same lines.

Theorem 3. By taking into account of (5), one needs only to show that the reverse inequality is true. For simplicity, let us consider again $r = 2$. Let $F(p^\alpha) = m$, $F(q^\beta) = n$ with $m \leq n$. By definition (1) one has $p^\alpha | f(m)$, $q^\beta | f(n)$. Now, by assumption (**) we can write $f(m) | f(n)$, so $p^\alpha | f(m) | f(n)$. Therefore, one has $p^\alpha | f(n)$, $q^\beta | f(n)$. This in turn implies $p^\alpha q^\beta | f(n)$, so $n \geq F(p^\alpha q^\beta)$; i.e. $\max\{F(p^\alpha), F(q^\beta)\} \geq F(p^\alpha q^\beta)$. The general case follows exactly the same lines. Thus, we have proved essentially, that $F(p^\alpha q^\beta) = \max\{F(p^\alpha), F(q^\beta)\}$, or more generally

$$F\left(\prod_{i=1}^r p_i^{\alpha_i}\right) = \max\{F(p_i^{\alpha_i}) : i = 1, 2, \dots, r\}. \quad (10)$$

Now, relation (7) is an immediate consequence of (10), for by writing

$$m = \prod_{i=1}^r p_i^{\alpha_i}, \quad n = \prod_{j=1}^s q_j^{\beta_j}, \quad \text{with } (p_i, q_j) = 1,$$

it follows that

$$\begin{aligned} F(mn) &= \max\{F(p_i^{\alpha_i}), F(q_j^{\beta_j}) : i = 1, 2, \dots, r, j = 1, 2, \dots, s\} \\ &= \max\{\max\{F(p_i^{\alpha_i}) : i = 1, 2, \dots, r\}, \max\{F(q_j^{\beta_j}) : j = 1, 2, \dots, s\}\} \\ &= \max\{F(m), F(n)\}, \end{aligned}$$

by equality (10).

Theorem 4. By (i) and definition (1) we get

$$F(n) \leq n. \quad (11)$$

Now, by (i), one has $p | f(p)$ for any prime p , but by (ii), p is the least such number. This implies that

$$F(p) = p \text{ for any prime } p. \quad (12)$$

Now, let k be an even perfect number. By the Euclid-Euler theorem (see e.g. [7]) k may be written as $k = 2^{n-1}(2^n - 1)$, where $p = 2^n - 1$ is a prime ("Mersenne prime"). Since (**) holds true, by Theorem 3 we can write

$$F(k) = F(2^{n-1}(2^n - 1)) = \max\{F(2^{n-1}), F(2^n - 1)\}.$$

Since $F(2^n - 1) = 2^n - 1$ (by (12)), and $F(2^{n-1}) \leq 2^{n-1}$ (by (11)), from $2^{n-1} < 2^n - 1$ for $n \geq 2$, we get $F(k) = 2^n - 1 = \frac{k}{2^s}$, where $s = n - 1$ and $2^s \parallel k$. This finishes the proof of Theorem 4.

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Some Smarandache conclusions of coloring properties of complete uniform mixed hypergraphs deleted some \mathcal{C} -hyperedges

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Abstract The upper and lower chromatic number of uniform mixed hypergraphs and \mathcal{C} -hyperedge and \mathcal{D} -hyperedge contact with the inevitable. In general, the increase in the \mathcal{C} -hyperedge will increase lower chromatic number $\chi_{\mathcal{H}}$, an increase \mathcal{D} -hyperedge will decrease upper chromatic number $\bar{\chi}_{\mathcal{H}}$. In this paper the relationship between \mathcal{C} -hyperedge with the upper chromatic number and lower chromatic number and some conclusions with respect to mixed hypergraph are given.

Keywords A complete uniform mixed hypergraph, chromatic, upper chromatic, Smarandache conclusions.

§1. Lemma and the basic concepts

Definition 1.1.^[1] Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set, $\mathcal{C} = \{C_1, C_2, \dots, C_l\}$, $\mathcal{D} = \{D_1, D_2, \dots, D_m\}$ are two subset clusters of X , all of which $C_i \in \mathcal{C}$ to meet $|C_i| \geq 2$, and all $D_j \in \mathcal{D}$ to meet $|D_j| \geq 2$. Then $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called a mixed hypergraph from X , and each $C_i \in \mathcal{C}$ is called the \mathcal{C} -hyperedges, and each $D_j \in \mathcal{D}$ is called the \mathcal{D} -hyperedges. In particular, $\mathcal{H}_D = (X, \mathcal{D})$ is called a \mathcal{D} hypergraph, the $\mathcal{H}_C = (X, \mathcal{C})$ for \mathcal{C} -hypergraph.

Definition 1.2.^[2] On $2 \leq l, m \leq n = |X|$, let

$$\mathcal{K}(n, l, m) = (X, \mathcal{C}, \mathcal{D}) = (X, \binom{X}{l}, \binom{X}{m})$$

where $|\mathcal{C}| = \binom{n}{l}$ and $|\mathcal{D}| = \binom{n}{m}$, then $\mathcal{K}(n, l, m)$ is called the complete (l, m) -uniform mixed hypergraph with n vertex.

It is clear that for a given n, l, m , in a sense of the isomorphic existence just has one $\mathcal{K}(n, l, m)$.

Definition 1.3.^[3,4] For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the largest i among all existence of strict i -Coloring known as the upper chromatic number \mathcal{H} , said that for $\bar{\chi}_{\mathcal{H}}$.

Definition 1.4.^[5] For mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if a i Partition $X = \{X_1, X_2, \dots, X_i\}$ of vertex sets X satisfy:

- 1) For each \mathcal{C} -hyperedge at least two vertices is allocated in the same block;
- 2) For each \mathcal{D} -hyperedge at least two vertices is allocated in different blocks.

The partition is called as a feasible partition of \mathcal{H} .

Obviously, any strict i coloring of \mathcal{H} corresponds with a strict i feasible partition, and vice versa. they are equivalent. Therefore, we write one feasible partition of \mathcal{H} or a strict i -coloring c as: $c = X_1 \cup X_2 \cup \dots \cup X_i$ and $r_i(\mathcal{H}) = r_i$ is the total number of all feasible i partition.

Definition 1.5.^[6] Let S be a subset of the vertices set X of mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if the set does not contain any of the \mathcal{C} -Hyperedge (\mathcal{D} -Hyperedge) as a subset, then it is called \mathcal{C} stable or \mathcal{C} independent (\mathcal{D} stable or \mathcal{D} independent).

Lemma 1.1.^[7,8] Let mixed hypergraph $\mathcal{H} = (X, \binom{X}{r}, \mathcal{D})$, where $2 \leq r \leq n = n(\mathcal{H})$, then Arbitrary a coloring of \mathcal{H} meet condition

$$\bar{\chi}(\mathcal{H}) = r - 1.$$

Definition 1.6.^[9] mixed hypergraph for $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, if there is a mapping $c : Y \rightarrow \{1, 2, \dots, \lambda\}$ that between subset $Y \in X$ and λ colors $\{1, 2, \dots, \lambda\}$, and it meet following conditions:

- 1) For each \mathcal{C} -hyperedge $C \in \mathcal{C}$, at least two vertices are the same color;
- 2) For each \mathcal{D} -hyperedge $D \in \mathcal{D}$, at least two vertices are different colors.

Then, mapping c is called as one λ colors normal coloring of the mixed hypergraph \mathcal{H} .

Definition 1.7.^[10] In a normal i -coloring of \mathcal{H} , if i colors are used, then the coloring is called a strict i -coloring.

It is clear that a normal $\chi(\mathcal{H})$ -coloring of mixed hypergraph \mathcal{H} must be a strict coloring.

Definition 1.8.^[11] For any coloring c of the mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, Let Y be a subset of X , then if Y satisfied: the arbitrary $y_1 \in Y, y_2 \in Y$, there is $c(y_1) = c(y_2)$, then we call the subset Y as monochrome; if each of two is different colors, that is, $c(y_1) \neq c(y_2)$, then we call subset Y as the multi-color.

By the definition the of the normal coloring of mixed hypergraph, we know that for any normal coloring of the hypergraph, \mathcal{D} -hyperedge not is a subset of monochrome, \mathcal{C} -hyperedge not is a subset of the multi-color.

Definition 1.9.^[12] In an arbitrary strictly i -coloring of \mathcal{H} , the vertex's set X of \mathcal{H} certainly is divided into i partition, each partition is a non-empty subset of monochrome, we call it as the color category.

Lemma 1.2.^[13] Let mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, and $n = |X|$, then regardless \mathcal{H} can normal coloring or not can, but the coloring of its sub-hypergraph $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ is always available and there is $\chi(\mathcal{H}_{\mathcal{C}}) = 1, r_1(\mathcal{H}_{\mathcal{C}}) = 1$ and $\bar{\chi}(\mathcal{H}_{\mathcal{D}}) = n(\mathcal{H}), r_n(\mathcal{H}_{\mathcal{D}}) = 1$.

Lemma 1.3.^[14] For mixed hypergraph $\mathcal{H}' = (X, \mathcal{C}, \binom{X}{m})$, if $\forall C \in \mathcal{C}$, where $|C| = k$ and $n(\mathcal{H}') \leq (k-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') \geq k-1$.

Lemma 1.4. Mixed hypergraph for $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, let \mathcal{H}' be arbitrary a subhypergraph of \mathcal{H} , the $\chi(\mathcal{H}) \geq \chi(\mathcal{H}')$, $\bar{\chi}(\mathcal{H}) \leq \bar{\chi}(\mathcal{H}')$.

Lemma 1.5.^[15] For a colorable mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$, where $2 \leq l, m \leq n$, then

- 1) $\chi(\mathcal{H}) = \lceil \frac{n(\mathcal{H})}{m-1} \rceil, \bar{\chi}(\mathcal{H}) = l-1$;
- 2) \mathcal{H} is the uncolorable if and only if $\lceil \frac{n(\mathcal{H})}{m-1} \rceil \geq l$.

Lemma 1.6.^[16] For \mathcal{C} hypergraph $\mathcal{H} = (X, \mathcal{C}, \emptyset)$, if $\forall C \in \mathcal{C}$ there are $|C| \geq k$, then $\bar{\chi}(\mathcal{H}) \geq k-1$, and all $(k-1)$ -coloring of \mathcal{H} are normal.

Lemma 1.7.^[17] If m articles put into n box, then at least one box contain not less than $\lceil \frac{m}{n} \rceil$ articles, where $\lceil \frac{m}{n} \rceil$ is not less than $\frac{m}{n}$ smallest integer.

Lemma 1.8.^[18] $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be anyone of the normal coloring of the mixed hypergraph \mathcal{H} , then

$$1 \leq \chi(\mathcal{H}_{\mathcal{D}}) \leq \chi(\mathcal{H}) \leq \bar{\chi}(\mathcal{H}) \leq \bar{\chi}(\mathcal{H}_{\mathcal{C}}) \leq n.$$

Lemma 1.9.^[19] For the mixed hypergraph $\mathcal{H}' = (X, \binom{X}{l} - \mathcal{C}', \binom{X}{m})$, where $\mathcal{C}' \subseteq \binom{X}{l}$, and $n = |X|$, then

- 1) When $n \leq (l-1)(m-1)$, \mathcal{H}' certainly at least has a normal $(l-1)$ coloring, that is, \mathcal{H}' certainly is colorable.
- 2) If \mathcal{H}' is the colorable, then $n \geq (l-1)(m-1) + 1$.

§2. The main results

Theorem. Let $\mathcal{H}' = (X, \binom{X}{l} - \mathcal{C}', \binom{X}{m})$ where $\mathcal{C}' \subseteq \binom{X}{l}$, and $|\mathcal{C}'| = k$, then

- 1) When $k = 1$, there are
 - I) If $|X| = l$, then $\bar{\chi}(\mathcal{H}') = l$;
 - II) If $l+1 \leq |X| \leq (l-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') = l-1$.
- 2) When $k = 2$, there are
 - I) If $|X| = l+1$, then $\bar{\chi}(\mathcal{H}') = l$;
 - II) If $l+2 \leq |X| \leq (l-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') = l-1$.
- 3) When $k \geq 3$, if $l+k-1 \leq |X| \leq (l-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') = l-1$.

Proof. 1) When $k = 1$, it is necessary to $|\binom{X}{l}| \geq 1$, so $|X| \geq l$. There are two cases as follows:

I) If $|X| = l$, then \mathcal{H}' does not contain \mathcal{C} -hyperedge. We can see the $\bar{\chi}(\mathcal{H}') = n(\mathcal{H}') = l$ from Lemma 1.2;

II) If $l+1 \leq |X| \leq (l-1)(m-1)$, By Lemma 1.6, $\bar{\chi}(\mathcal{H}') \geq l-1$.

As long as prove $\bar{\chi}(\mathcal{H}') \leq l-1$, that's all.

Use reduction to absurdity. If not, that is $\bar{\chi}(\mathcal{H}') \geq l$. there are two cases.

2.1. If $\bar{\chi}(\mathcal{H}') \geq l+1$, then there is a feasible $\bar{\chi}(\mathcal{H}')$ partition of \mathcal{H}' , Let $X = X_1 \cup X_2 \cup \dots \cup X_{\bar{\chi}(\mathcal{H}')}$, where each $X_i (i = 1, 2, \dots, \bar{\chi}(\mathcal{H}'))$ non-empty, And $X_i \cap X_j = \emptyset (i \neq j)$, then there is $\bar{\chi}(\mathcal{H}')$ separate vertex $x_i \in X_i (i = 1, 2, \dots, \bar{\chi}(\mathcal{H}'))$, makes $\{x_1, x_2, \dots, x_{\bar{\chi}(\mathcal{H}')} \}$ for a \mathcal{C} -stability set. Thus $\bar{\chi}(\mathcal{H}')$ vertex does not form any \mathcal{C} -hyperedge, so, $|\mathcal{C}'| \geq (\bar{\chi}(\mathcal{H}'))$.

We also available from the assumption that: $|\mathcal{C}'| \geq \binom{l+1}{l} = l+1 > 2$, this is contradictory with $|\mathcal{C}'| = 1$. So, $\bar{\chi}(\mathcal{H}') \geq l+1$ is impossible.

2.2. If $\bar{\chi}(\mathcal{H}') = l$, then there is a feasible l partition of \mathcal{H}' , let $X = X_1 \cup X_2 \cup \dots \cup X_l$, as a result of $l+1 \leq |X| \leq (l-1)(m-1)$, therefore, by the lemma 1.7, at least a color category contains at least $\lceil \frac{|X|}{l} \rceil \geq \lceil \frac{l+1}{l} \rceil = 2$ vertices. assume it is X_1 . As the others of the color classes are non-empty, so certainly there are $l-1$ different points: $x_i \in X_i (i = 2, 3, \dots, l)$, they together with $x_1, x'_1 \in X_1$ formate two \mathcal{C} -stability sets $\{x_1, x_2, \dots, x_l\}$ and $\{x'_1, x_2, \dots, x_l\}$ For \mathcal{H}' . That is, \mathcal{H}' at least does not contain them. So, $|\mathcal{C}'| \geq 2$. This with the known condition $|\mathcal{C}'| = 1$ contradictions. So $\bar{\chi}(\mathcal{H}') \neq l$. By case 2.1 and 2.2, we can see that $\bar{\chi}(\mathcal{H}') \leq l-1$.

Overview on, we can see that when $k = 1$, if $l+1 \leq |X| \leq (l-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') = l-1$.

2) When $k = 2$, it is necessary to $|\binom{X}{l}| \geq 2$, so $|X| \geq l+1$. There are two cases.

Case 1. If $|X| = l+1$, Let $X = \{x_1, x_2, \dots, x_l, x_{l+1}\}$, then two \mathcal{C} -hyperedge in \mathcal{C}' certainly have $l-1$ vertices are the same, that is, and there is only one Vertex is not the same. Let they are $C_1 = \{x_1, x_2, \dots, x_l\}$ and $C_2 = \{x_2, \dots, x_{l+1}\}$. Use the following coloring method for \mathcal{H}' coloring: Allocate color 1 for vertex x_1 and x_{l+1} , the remaining $l-1$ vertices x_2, \dots, x_l separately allocate colors $2, 3, \dots, l$, clearly, this is a strict l coloring of \mathcal{H}' , therefore $\bar{\chi}(\mathcal{H}') \geq l$.

Following prove that $\bar{\chi}(\mathcal{H}') \leq l$.

Use reduction to absurdity. If not, then $\bar{\chi}(\mathcal{H}') \geq l+1$. For easily state, we let $\bar{\chi}(\mathcal{H}') = k$. as a result of $|X| = l+1$, Therefore, by the lemma 1.7 and lemma 1.8 we can see $k \leq l+1$, then only is $k = l+1$. As a result, certainly, there is a feasible $l+1$ Partition of \mathcal{H}' , let it is $X = X_1 \cup X_2 \cup \dots \cup X_{l+1}$, where $X_i \neq \emptyset (i = 1, 2, \dots, l+1)$, and $X_i \cap X_j = \emptyset (i \neq j)$. Thus, there is $l+1$ different vertices $x_i \in X_i (i = 1, 2, \dots, l+1)$, they formate set $\{x_1, x_2, \dots, x_{l+1}\}$, clearly, this is a \mathcal{C} -stability set of \mathcal{H}' . That is, $\{x_1, x_2, \dots, x_{l+1}\}$ does not contain any \mathcal{C} -hyperedge, then, $|\mathcal{C}'| \geq \binom{l+1}{l} = l+1 > 2$, This is in contradiction with the known conditions. Therefore, $k \neq l+1$. So that $\bar{\chi}(\mathcal{H}') \leq l$.

Overview above, we can see that when $k = 2$, if $|X| = l+1$, then $\bar{\chi}(\mathcal{H}') = l$.

Cases 2. If $l+2 \leq |X| \leq (l-1)(m-1)$, we can see from Lemma 1.3 $\bar{\chi}(\mathcal{H}') \geq l-1$. As long as proved that $\bar{\chi}(\mathcal{H}') \leq l-1$, that's all.

If not, then $\bar{\chi}(\mathcal{H}') \geq l$. For the convenience of expression, we let that $\bar{\chi}(\mathcal{H}') = s$, from the assumption we easy know that $s \geq l$. there are two circumstances under which says:

Case 2.1. If $s = l$, by the definition of the strict normal coloring and the feasible partition, we can see that certainly there is at least one the feasible l Partition of \mathcal{H}' , let it is $X = X_1 \cup X_2 \cup \dots \cup X_l$, where $X_i \neq \emptyset (i = 1, 2, \dots, l)$ and $X_i \cap X_j = \emptyset (i \neq j)$.

Because $l+2 \leq |X| \leq (l-1)(m-1)$, therefore, the partition $X = X_1 \cup X_2 \cup \dots \cup X_l$ may be the only the following two circumstances:

- i) At least one color class include at least with more than 3 vertices, and the remaining color classes include at least a vertex;
- ii) At least two color classes include at least two or more vertices, and the remaining color classes include at least a vertex.

When the case *i*), we let color class X_1 include at least more than 3 vertices and the rest include at least a vertex, and let $x_1, x'_1, x''_1 \in X_1, x_2 \in X_2, \dots, x_l \in X_l$. by the definition of feasible partition, we know that $\{x_1, x_2, \dots, x_l\}, \{x'_1, x_2, \dots, x_l\}$ and $\{x''_1, x_2, \dots, x_l\}$ are three \mathcal{C} -stability Sets of mixed hypergraph \mathcal{H}' . Then \mathcal{H}' at least does not contain they. that is, they are included by \mathcal{C}' , So, $|\mathcal{C}'| \geq 3$, This contradictions with known condition $|\mathcal{C}'| = 2$, so $s \neq l$.

When the case *ii*), we let that color class X_1 and X_2 contain at least two or more vertices, and the remaining color classes contain at least a vertex, Let $x_1, x'_1 \in X_1, x_2, x'_2 \in X_2, x_3 \in X_3, \dots, x_l \in X_l$. by the definition of the feasible partition, we know that $\{x_1, x_2, \dots, x_l\}, \{x_1, x'_2, \dots, x_l\}, \{x'_1, x_2, \dots, x_l\}$ and $\{x'_1, x'_2, \dots, x_l\}$ are 4 \mathcal{C} -stability set of the mixed hypergraph \mathcal{H}' . That is, \mathcal{H}' at least does not contain them. So, $|\mathcal{C}'| \geq 4$, This contradictions with known condition $|\mathcal{C}'| = 2$. so $s \neq l$ under this circumstances.

Overview above, we know that if $l+2 \leq |X| \leq (l-1)(m-1)$, then $s \neq l$.

Case 2.2. If $s \geq l+1$, by the definition of the strict normal coloring and the feasible parti-

tion, we know that certainly there is at least one the feasible s Partition $X = X_1 \cup X_2 \cup \cdots \cup X_s$ of \mathcal{H}' , where $X_i \neq \emptyset (i = 1, 2, \cdots, s)$ and $X_i \cap X_j = \emptyset (i \neq j)$, then exist s different from each other vertices $x_i \in X_i (i = 1, 2, \cdots, s)$, makes $\{x_1, x_2, \cdots, x_s\}$ for the \mathcal{C} -stability set of \mathcal{H}' . That is, $\{x_1, x_2, \cdots, x_s\}$ dose not contain any \mathcal{C} -hyperedge, so $|\mathcal{C}'| \geq \binom{s}{l} \geq \binom{l+1}{l} = l+1 > 2$, this contradictions with known conditions $|\mathcal{C}'| = k = 2$. So, assumption condition $s \geq l+1$ does not hold.

Integrated proven of cases 2.1 and cases 2.2 we know that $\bar{\chi}(\mathcal{H}') \leq l-1$.

As a result, when $k = 2$, if $l+2 \leq |X| \leq (l-1)(m-1)$, then $\bar{\chi}(\mathcal{H}') = l-1$.

Through the proved course of case 1 and case 2, we know that the conclusion of the (2) is proper.

3) When $k \geq 3$, if $l+k-1 \leq |X| \leq (l-1)(m-1)$, by the conclusion 1) of a lemma 1.9, we know that \mathcal{H}' is colorable, and by Lemma 1.3 we know that the $\bar{\chi}(\mathcal{H}') \geq l-1$.

We prove $\bar{\chi}(\mathcal{H}') \leq l-1$, that's all.

Still we use the reduction to absurdity. If not, assume that $\bar{\chi}(\mathcal{H}') \geq l$. For described be convenient, we let $\bar{\chi}(\mathcal{H}') = r$.

Through the definition of the Strictly normal coloring and the feasible partition, we know that there is a certain, the feasible r Partition of \mathcal{H}' : $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where $X_i \neq \emptyset (i = 1, 2, \cdots, r)$, $X_i \cap X_j = \emptyset (i \neq j)$.

As each color classes is non-empty, so there is $x_i \in X_i (i = 1, 2, \cdots, r)$, and because $|X| \geq l+k-1$,

Therefore, in addition to the X contain x_1, x_2, \cdots, x_r these r vertices, It also includes at least $l+k-1-r$ vertices: $x'_1, x'_2, \cdots, x'_{(l+k-1-r)}$. The $l+k-1-r$ vertices arbitrarily are assigned to above r color classes, possible different assigned methods only are the following cases:

Case 1. Above $l+k-1-r$ vertices were separately assigned to the $l+k-1-r$ the color classes;

Case 2. They were assigned to a total of $l+k-2-r$ the color classes;

...

Case $l+k-1-r$. They were assigned to the same color class.

Before study these cases, we prove the following assertion:

Assertion. Let $s = |\{Y | Y \text{ is } \mathcal{C}\text{-stability Set of mixed hypergraph } \mathcal{H}', \text{ and } |Y| \geq l\}|$, in above all the $l+k-1-r$ different cases, the $l+k-1-r$ kind of cases correspond s value is the minimum among all.

Proof. To make the following agreement: For any the feasible r Partition $X = X_1 \cup X_2 \cup \cdots \cup X_r$ of the \mathcal{H}' , We assume $|X_1| \geq |X_2| \geq \cdots \geq |X_r|$. Obviously, the s value only is related with the vertex's number of each color class, and is not related with the order of all color classes arrayed. Thus under the condition of keep the s value unchange, through exchange color classes order, we can do this. In this agreement, According to above the $l+k-1-r$ kind of all circumstances Distributed $x'_1, x'_2, \cdots, x'_{(l+k-1-r)}$ to each color class, then the feasible r Partition of correspondence with this kind distributed certainly meet to the following condition:

$|X_1| \geq l+k-r, |X_i| \geq 1 (i = 2, 3, \cdots, r)$, we may let $|X_1| = n_1, |X_2| = n_2, \cdots, |X_r| = n_r$, then the s value of corresponding to this feasible partition is:

$$\begin{aligned}
s_0 &= \sum_{i=l}^r \sum_{\substack{\{j_1, j_2, \dots, j_i\} \subseteq \{1, 2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_i}| \\
&= \sum_{i=l}^r [|X_1| \sum_{\substack{\{j_1, j_2, \dots, j_{i-1}\} \subseteq \{2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_{i-1}}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_{i-1}}| \\
&\quad + \sum_{\substack{\{j'_1, j'_2, \dots, j'_i\} \subseteq \{2, \dots, r\} \\ |X_{j'_1}| \geq |X_{j'_2}| \geq \dots \geq |X_{j'_i}|}} |X_{j'_1}| |X_{j'_2}| \cdots |X_{j'_i}|] \\
&\geq (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}. \tag{1}
\end{aligned}$$

It is easy to know that $(l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}$ precisely represent to the s value corresponding with the feasible partition $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where $x_i \in X_i (i = 1, 2, \dots, r)$, $x'_j \in X_1 (j = 1, 2, \dots, l+k-1-r)$, this the feasible partition is attained through allocate $x_1, x_2, \dots, x_r, x'_1, x'_2, \dots, x'_{(l+k-1-r)}$ to each color class according to above the $l+k-1-r$ kinds of cases, and when $X = \{x_1, x_2, \dots, x_r, x'_1, x'_2, \dots, x'_{(l+k-1-r)}\}$, That is, when $n(\mathcal{H}') = l+k-1$.

Combination formula (1), we know that the s value when $n(\mathcal{H}') > l+k-1$ is greater than when $n(\mathcal{H}') = l+k-1$. thus, in order to make access to the minimum s , have to be $|X| = n(\mathcal{H}') = l+k-1$.

The following as further prove that when $|X| = l+k-1$, only according to above the $l+k-1-r$ kind of cases allocate vertices to each color class, corresponding s value is the minimum.

In accordance with the above-mentioned in article, we let $X = \{x_1, x_2, \dots, x_r, x'_1, x'_2, \dots, x'_{(l+k-1-r)}\}$, and let $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where $x_i \in X_i (i = 1, 2, \dots, r)$, $x'_j \in X_1 (j = 1, 2, \dots, l+k-1-r)$, is a feasible partition corresponding with above the $l+k-1-r$ kind of cases. then other cases corresponding to the feasible partition can be seen as is according to following method obtained:

Selecting out some vertices from color class X_1 , then put them in other color classes, and keep the relationship $|X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|$.

Let the vertices number which selected out from X_1 for p , following through mathematical induction on p to prove that assertion.

When $p = 1$, we may let selected out from X_1 vertex for $x'_{(l+k-1-r)}$, by above the agreement, we know that the feasible partition of re-distributed the vertex $x_{(l+k-1-r)}$ only may is: $X_1 = \{x_1, x'_1, x'_2, \dots, x'_{(l+k-2-r)}\}$, $X_2 = \{x_2, x'_{(l+k-2-r)}\}$, $X_i = \{x_i\} (i = 3, \dots, r)$, clearly, corresponding with it s value is:

$$\begin{aligned}
&\sum_{i=l}^r \sum_{\substack{\{j_1, j_2, \dots, j_i\} \subseteq \{1, 2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_i}| \\
&= |X_1| \sum_{i=l}^r \binom{r-2}{i-1} + |X_2| \sum_{i=l}^r \binom{r-2}{i-1} + |X_1| |X_2| \sum_{i=l}^r \binom{r-2}{i-2} + \sum_{i=l}^r \binom{r-2}{i} \\
&= [(l+k-r) + 2] \sum_{i=l}^r \binom{r-2}{i-1} + 2(l+k-r) \sum_{i=l}^r \binom{r-2}{i-2} + \sum_{i=l}^r \binom{r-2}{i}.
\end{aligned}$$

In addition, it is clear that the $l+k-1-r$ kinds of cases corresponding to the feasible Partition $X = X_1 \cup X_2 \cup \cdots \cup X_r$, where $x_i \in X_i (i = 1, 2, \cdots, r)$, $x'_j \in X_1 (j = 1, 2, \cdots, l+k-1-r)$ corresponding s value is:

$$\begin{aligned} & \sum_{i=l}^r \sum_{\substack{\{j_1, j_2, \dots, j_i\} \subseteq \{1, 2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_i}| \\ &= |X_1| \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i} \\ &= (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}. \end{aligned}$$

It is clear that

$$\begin{aligned} & [(l+k-r) + 2] \sum_{i=l}^r \binom{r-2}{i-1} \geq (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1}, \\ & 2(l+k-r) \sum_{i=l}^r \binom{r-2}{i-2} + \sum_{i=l}^r \binom{r-2}{i} \geq \sum_{i=l}^r \binom{r-1}{i}, \end{aligned}$$

As a result,

$$\begin{aligned} & [(l+k-r) + 2] \sum_{i=l}^r \binom{r-2}{i-1} + 2(l+k-r) \sum_{i=l}^r \binom{r-2}{i-2} + \sum_{i=l}^r \binom{r-2}{i} \\ & \geq (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}. \end{aligned}$$

That is, when $p = 1$, the conclusion is true.

Assumption that when $p < t$ the conclusions also are true. Then when $p = t$, that is, from the color class X_1 select out t vertices re-assigned to other color classes and to maintain relations $|X_{j_1}| \geq |X_{j_2}| \geq \cdots \geq |X_{j_i}|$.

We may let that the t time selected out from X_1 vertex is $x'_{(l+k-t-r)}$, and it will be re-assigned to the color class X_j . And let this color classes $X_2, \cdots, X_{j-1}, X_j, X_{j+1}, \cdots, X_r$ contained in the vertex total number different were: $n_2, \cdots, n_{j-1}, n_j, n_{j+1}, \cdots, n_r$, at the same time we let that this step operation before the color class X_1 contained Vertices number for $l+k-r-t+1$, then after the operation of this step each color class contained vertex number different are $|X_1| = l+k-r-t$, $|X_2| = n_2, \cdots, |X_{j-1}| = n_{j-1}, |X_j| = n_j + 1, |X_{j+1}| = n_{j+1}, \cdots, |X_r| = n_r$.

It is easy to know s value corresponding with the feasible partition that obtained by after the $t-1$ step operation is:

$$\begin{aligned}
s_{t-1} &= \sum_{i=l}^r \sum_{\substack{\{j_1, j_2, \dots, j_i\} \subseteq \{1, 2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_i}| \\
&= \sum_{i=l}^r \left[(l+k-t+1) \sum_{\substack{\{j_1, j_2, \dots, j_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j_1 < j_2 < \dots < j_{i-1}}} n_{j_1} n_{j_2} \cdots n_{j_{i-1}} \right. \\
&\quad + n_j \sum_{\substack{\{j'_1, j'_2, \dots, j'_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'_1 < j'_2 < \dots < j'_{i-1}}} n_{j'_1} n_{j'_2} \cdots n_{j'_{i-1}} \\
&\quad + (l+k-t+1)n_j \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&\quad \left. + \sum_{\substack{\{j'''_1, j'''_2, \dots, j'''_i\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'''_1 < j'''_2 < \dots < j'''_i}} n_{j'''_1} n_{j'''_2} \cdots n_{j'''_i} \right] \\
&= \sum_{i=l}^r \left[(l+k-t+n_j+1) \sum_{\substack{\{j_1, j_2, \dots, j_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j_1 < j_2 < \dots < j_{i-1}}} n_{j_1} n_{j_2} \cdots n_{j_{i-1}} \right. \\
&\quad + (l+k-t+1)n_j \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&\quad \left. + \sum_{\substack{\{j'''_1, j'''_2, \dots, j'''_i\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'''_1 < j'''_2 < \dots < j'''_i}} n_{j'''_1} n_{j'''_2} \cdots n_{j'''_i} \right].
\end{aligned}$$

By the assumption of mathematical induction we know that $s_{t-1} \geq (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}$

Therefore, the s value corresponding to the feasible partition of that after the T step operation is:

$$\begin{aligned}
s_t &= \sum_{i=l}^r \sum_{\substack{\{j_1, j_2, \dots, j_i\} \subseteq \{1, 2, \dots, r\} \\ |X_{j_1}| \geq |X_{j_2}| \geq \dots \geq |X_{j_i}|}} |X_{j_1}| |X_{j_2}| \cdots |X_{j_i}| \\
&= \sum_{i=l}^r \left[(l+k-t) \sum_{\substack{\{j_1, j_2, \dots, j_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j_1 < j_2 < \dots < j_{i-1}}} n_{j_1} n_{j_2} \cdots n_{j_{i-1}} \right. \\
&\quad + (n_j + 1) \sum_{\substack{\{j'_1, j'_2, \dots, j'_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'_1 < j'_2 < \dots < j'_{i-1}}} n_{j'_1} n_{j'_2} \cdots n_{j'_{i-1}} \\
&\quad + (l+k-t)(n_j + 1) \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&\quad \left. + \sum_{\substack{\{j'''_1, j'''_2, \dots, j'''_i\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'''_1 < j'''_2 < \dots < j'''_i}} n_{j'''_1} n_{j'''_2} \cdots n_{j'''_i} \right] \\
&= \sum_{i=l}^r \left[(l+k-t+n_j+1) \sum_{\substack{\{j_1, j_2, \dots, j_{i-1}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j_1 < j_2 < \dots < j_{i-1}}} n_{j_1} n_{j_2} \cdots n_{j_{i-1}} \right. \\
&\quad + (l+k-t+1)n_j \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&\quad \left. + \sum_{\substack{\{j'''_1, j'''_2, \dots, j'''_i\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j'''_1 < j'''_2 < \dots < j'''_i}} n_{j'''_1} n_{j'''_2} \cdots n_{j'''_i} \right] \\
&\quad + (l+k-t-n_j) \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&= s_{t-1} + (l+k-t-n_j) \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}} \\
&\geq (l+k-r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i} \\
&\quad + (l+k-t-n_j) \sum_{\substack{\{j''_1, j''_2, \dots, j''_{i-2}\} \subseteq \{2, \dots, j-1, j+1, \dots, r\} \\ j''_1 < j''_2 < \dots < j''_{i-2}}} n_{j''_1} n_{j''_2} \cdots n_{j''_{i-2}}.
\end{aligned}$$

According to the agreement given when begin prove, after t step operation, the vertices' number which contained in color class X_1 is $l+k-r-t$, and it is not less than that any other color classes contained, namely $l+k-r-t \geq n_j+1$. Then $l+k-t-n_j \geq t+1 > 0$. Therefore,

the feasible partition of that obtained after the t step operation corresponding value of s is

$$s_t \geq (l + k - r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}.$$

That is, when $p = t$, the conclusions are true.

Overview above, we know that the conclusions are true for any natural number p .

In the following we prove this theorem conclusions on the basis of above assertion, It is clear that by the assertion we the easy access to the following conclusions:

1) When $\bar{\chi}(\mathcal{H}') = l$, by the assertion we know that when the $n(\mathcal{H}') = l + k - 1$, s value is the minimum and $s_{min} = k$; if the vertices number $n(\mathcal{H}') > l + k - 1$, then $s > k$.

2) When $n(\mathcal{H}') \geq l + k - 1$ and it is a definite value, and $\bar{\chi}(\mathcal{H}') > l$, by above assertion, We know that the value of s corresponding to this condition certainly greater than k .

Combination of above two conclusions, we know that when $l + k - 1 \leq |X| \leq (l - 1)(m - 1)$, $\bar{\chi}(\mathcal{H}') \geq l$ must have $s > k$.

Again we return to the theorem proof. By the assumption $\bar{\chi}(\mathcal{H}') \geq l$ of that given when begin prove conclusions from 3) and the known conditions $l + k - 1 < |X| \leq (l - 1)(m - 1)$, and combination proved just conclusion, we know $s > k$.

Because s represents the total number of \mathcal{C} -stable set that is contained from one the feasible partition corresponding to a strict $\bar{\chi}(\mathcal{H}') \geq l$ coloring of \mathcal{H}' , and the cardinal number of \mathcal{C} -stable set is greater than l or equal to l . Because, each \mathcal{C} -stable set which the cardinal number of set is greater than l or equal to l contain at least one \mathcal{C} -hyperedge of the mixed hypergraph $\mathcal{H} = (X, \binom{X}{l}, \binom{X}{m})$. so mixed hypergraph \mathcal{H}' should does not contain at least s \mathcal{C} -hyperedge, that is, $|\mathcal{C}'| \geq s > k$, with known conditions $|\mathcal{C}'| = k$ contradictions. Then, the assume is untrue. Therefore, $\bar{\chi}(\mathcal{H}') \leq l - 1$.

By above the whole process of prove, we know that when $k \geq 3$, if $l + k - 1 < |X| \leq (l - 1)(m - 1)$, then $\bar{\chi}(\mathcal{H}') = l - 1$.

Corollary. Let $\mathcal{H}' = (X, \binom{X}{l} - \mathcal{C}', \binom{X}{m})$, where $\mathcal{C}' \subseteq \binom{X}{l}$, and let $|\mathcal{C}'| = k$, and s be the number of \mathcal{C} -stable set of \mathcal{H}' , then

$$s_{min} = (l + k - r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i} + \sum_{i=1}^{l-1} \binom{n}{i}$$

where $n = |X|$.

Proof. \mathcal{C} -stable set of \mathcal{H}' is divided into two groups: one is those the cardinal number less than l ; the other is those the cardinal number not less than l . By the above assertion and its proof we know that the total number of \mathcal{C} -stable set those cardinal number not less than l at least equal to $(l + k - r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i}$. It is easy obtain that total number of \mathcal{C} -stable set those the cardinal number less than l is $\sum_{i=1}^{l-1} \binom{n}{i}$, so,

$$s_{min} = (l + k - r) \sum_{i=l}^r \binom{r-2}{i-1} + \sum_{i=l}^r \binom{r-1}{i} + \sum_{i=1}^{l-1} \binom{n}{i}.$$

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A limit problem of the Smarandache dual function $S^{**}(n)$ ¹

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Abstract For any positive integer n , the Smarandache dual function $S^{**}(n)$ is defined as

$$S^{**}(n) = \begin{cases} \max \{2m : m \in N^*, (2m)!! \mid n\}, & 2 \mid n; \\ \max \{2m-1 : m \in N^*, (2m-1)!! \mid n\}, & 2 \nmid n. \end{cases}$$

The main purpose of this paper is using the elementary methods to study the convergent properties of an infinity series involving $S^{**}(n)$, and give an interesting limit formula for it.

Keywords The Smarandache dual function, limit problem, elementary method.

§1. Introduction and Results

For any positive integer n , the Smarandache dual function $S^{**}(n)$ is defined as the greatest positive integer $2m-1$ such that $(2m-1)!!$ divide n , if n is an odd number; $S^{**}(n)$ is the greatest positive $2m$ such that $(2m)!!$ divides n , if n is an even number. From the definition of $S^{**}(n)$ we know that the first few values of $S^{**}(n)$ are: $S^{**}(1) = 1$, $S^{**}(2) = 2$, $S^{**}(3) = 3$, $S^{**}(4) = 2$, $S^{**}(5) = 1$, $S^{**}(6) = 2$, $S^{**}(7) = 1$, $S^{**}(8) = 4$, \dots . About the elementary properties of $S^{**}(2)$, some authors had studied it, and obtained many interesting results. For example, Su Gou [1] proved that for any real number $s > 1$, the series $\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}$ is absolutely convergent, and

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s} = \zeta(s) \left(1 - \frac{1}{2^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{2}{((2m+1)!!)^s}\right) + \zeta(s) \left(\sum_{m=1}^{\infty} \frac{2}{((2m)!!)^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

Yanting Yang [2] studied the mean value estimate of $S^{**}(n)$, and gave an interesting asymptotic formula:

$$\sum_{n \leq x} S^{**}(n) = x \left(2e^{\frac{1}{2}} - 3 + 2e^{\frac{1}{2}} \int_0^1 e^{-\frac{y^2}{2}} dy\right) + O(\ln^2 x),$$

where $e = 2.7182818284 \dots$ is a constant.

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Yang Wang [3] also studied the mean value properties of $S^{**}(n)^2$, and prove that

$$\sum_{n \leq x} S^{**}(n)^2 = \frac{13x}{2} + O\left(\left(\frac{\ln x}{\ln \ln x}\right)^3\right).$$

In this paper, we using the elementary method to study the convergent properties of the series

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s},$$

and give an interesting identity and limit theorem. That is, we shall prove the following:

Theorem. For any real number $s > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \zeta(s) \left[1 - \frac{1}{2^s} + \left(1 - \frac{1}{2^s}\right) \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} + \sum_{m=1}^{\infty} \frac{8m-4}{((2m)!!)^s} \right],$$

where $\zeta(s)$ is the Riemann zeta-function.

From this Theorem we may immediately deduce the following limit formula:

Corollary. We have the limit

$$\lim_{s \rightarrow 1} (s-1) \left(\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} \right) = \frac{13}{2}.$$

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem directly. It is clear that $S^{**}(n) \ll \ln n$, so if $s > 1$, then the series $\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s}$ is convergent absolutely, so we have

$$\sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} + \sum_{\substack{n=1 \\ 2 \mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s} \equiv S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}, \quad S_2 = \sum_{\substack{n=1 \\ 2 \mid n}}^{\infty} \frac{S^{**}(n)^2}{n^s}.$$

From the definition of $S^{**}(n)$ we know that if $2 \nmid n$, we can assume that $S^{**}(n) = 2m-1$, then $(2m-1)!! \mid n$. Let $n = (2m-1)!!u$, $2m+1 \nmid u$. Note that the identity

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \left(1 - \frac{1}{2^s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 - \frac{1}{2^s}\right) \zeta(s),$$

so from the definition of $S^{**}(n)$ we can deduce that ($s > 1$),

$$\begin{aligned}
 S_1 &= \sum_{m=1}^{\infty} \sum_{\substack{u=1, 2 \nmid u \\ 2m+1 \nmid u}}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s u^s} \\
 &= \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} \sum_{\substack{u=1, 2 \nmid u \\ 2m+1 \nmid u}}^{\infty} \frac{1}{u^s} \\
 &= \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} \left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \frac{1}{(2m+1)^s} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \right) \\
 &= \zeta(s) \left(1 - \frac{1}{2^s} \right) \left(\sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m-1)!!)^s} - \sum_{m=1}^{\infty} \frac{(2m-1)^2}{((2m+1)!!)^s} \right) \\
 &= \zeta(s) \left(1 - \frac{1}{2^s} \right) \left(1 + \sum_{m=1}^{\infty} \frac{(2m+1)^2 - (2m-1)^2}{((2m+1)!!)^s} \right) \\
 &= \zeta(s) \left(1 - \frac{1}{2^s} \right) \left(1 + \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} \right).
 \end{aligned}$$

For even number n , we assume that $S^{**}(n) = 2m$, then $(2m)!! \mid n$. Let $n = (2m)!!v$, $2m+2 \nmid v$. If $s > 1$, then we can deduce that

$$\begin{aligned}
 S_2 &= \sum_{m=1}^{\infty} \sum_{\substack{v=1 \\ 2m+2 \nmid v}}^{\infty} \frac{(2m)^2}{((2m)!!)^s v^s} \\
 &= \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} \sum_{\substack{v=1 \\ (2m+2) \nmid v}}^{\infty} \frac{1}{v^s} \\
 &= \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} \left(\sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{(2m+2)^s} \sum_{n=1}^{\infty} \frac{1}{n^s} \right) \\
 &= \zeta(s) \left(\sum_{m=1}^{\infty} \frac{(2m)^2}{((2m)!!)^s} - \sum_{m=1}^{\infty} \frac{(2m)^2}{((2m+2)!!)^s} \right) \\
 &= \zeta(s) \left(\frac{1}{2^{s-2}} + \sum_{m=1}^{\infty} \frac{(2m+2)^2 - (2m)^2}{((2m+2)!!)^s} \right) \\
 &= \zeta(s) \left(\frac{1}{2^{s-2}} + \sum_{m=1}^{\infty} \frac{8m+4}{((2m+2)!!)^s} \right) \\
 &= 4\zeta(s) \sum_{m=1}^{\infty} \frac{2m-1}{((2m)!!)^s}.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{S^{**}(n)^2}{n^s} &= S_1 + S_2 \\
 &= \zeta(s) \left(1 - \frac{1}{2^s}\right) \left(1 + \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s}\right) + 4\zeta(s) \sum_{m=1}^{\infty} \frac{2m-1}{((2m)!!)^s} \\
 &= \zeta(s) \left[1 - \frac{1}{2^s} + \left(1 - \frac{1}{2^s}\right) \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} + \sum_{m=1}^{\infty} \frac{8m-4}{((2m)!!)^s}\right].
 \end{aligned}$$

This completes the proof of our Theorem.

Now we prove Corollary, note that

$$\begin{aligned}
 &\frac{1}{2} + \sum_{m=1}^{\infty} \frac{4m}{(2m+1)!!} + \sum_{m=1}^{\infty} \frac{8m-4}{(2m)!!} \\
 &= \frac{1}{2} + \sum_{m=1}^{\infty} \left(\frac{2}{(2m-1)!!} - \frac{2}{(2m+1)!!}\right) + \sum_{m=1}^{\infty} \left(\frac{4}{(2m-2)!!} - \frac{4}{(2m+2)!!}\right) \\
 &= \frac{1}{2} + 2 + 4 = \frac{13}{2}
 \end{aligned}$$

and

$$\lim_{s \rightarrow 1} (s-1)\zeta(s) = 1,$$

from Theorem we may immediately deduce that

$$\begin{aligned}
 &\lim_{s \rightarrow 1} (s-1) \left(\sum_{n=1}^{\infty} \frac{S^{**}(n)}{n^s}\right) \\
 &= \lim_{s \rightarrow 1} (s-1)\zeta(s) \left[1 - \frac{1}{2^s} + \left(1 - \frac{1}{2^s}\right) \sum_{m=1}^{\infty} \frac{8m}{((2m+1)!!)^s} + \sum_{m=1}^{\infty} \frac{8m-4}{((2m)!!)^s}\right] \\
 &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{4m}{(2m+1)!!} + \sum_{m=1}^{\infty} \frac{8m-4}{(2m)!!} = \frac{13}{2}.
 \end{aligned}$$

This completes the proof of Corollary.

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Separation axioms in smooth fuzzy topological spaces

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Abstract In this paper, the concepts of fuzzy $\tilde{g}-T_i$ ($i = 0, 1, 2, 1/2$) spaces, fuzzy \tilde{g} -normal spaces and fuzzy \tilde{g} -regular spaces in the sense of Sostak [7] are introduced. Also, some interesting properties and characterizations of them are investigated.

Keywords Fuzzy $\tilde{g}-T_i$ ($i = 0, 1, 2, 1/2$) spaces, fuzzy \tilde{g} -normal spaces, fuzzy \tilde{g} -regular spaces.

§1. Introduction and Preliminaries

The concept of fuzzy set was introduced by Zadeh [10] in his classical paper. Fuzzy sets have applications in many fields such as information [6] and control [8]. In 1985, Sostak [7] introduced a new form of topological structure. In 1992, Ramadan [4] studied the concept of smooth fuzzy topological spaces. The concept of \tilde{g} -open set was discussed by Rajesh and Erdal Ekici [3]. The concept of fuzzy normal spaces was introduced by Bruce Hutton [1]. Kubiak [9] established many interesting properties of fuzzy normal spaces. The purpose of this paper is to introduce fuzzy $\tilde{g}-T_i$ ($i = 0, 1, 2, 1/2$) spaces, \tilde{g} -normality and \tilde{g} -regularity in smooth fuzzy topological spaces. Also many interesting characterizations are established.

Throughout this paper, let X be a nonempty set, $I = [0, 1]$ and $I_0 = (0, 1]$. For $\langle \in I$, $T(x) = \langle$ for all $x \in X$.

Definition 1.1. [2] A fuzzy point x_t in X is a fuzzy set taking value $t \in I_0$ at x and zero elsewhere, $x_t \in \lambda$ iff $t \leq \lambda(x)$. A fuzzy set λ is quasi-coincident with a fuzzy set μ , denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise $\lambda \bar{q} \mu$.

Definition 1.2. [7] A function $T : I^X \rightarrow I$ is called a smooth fuzzy topology on X if it satisfies the following conditions :

- (1) $T(\bar{0}) = T(\bar{1}) = 1$;
- (2) $T(\mu_1 \wedge \mu_2) \geq T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.
- (3) $T(\bigvee_{j \in \Gamma} \mu_j) \geq \bigwedge_{j \in \Gamma} T(\mu_j)$ for any $\{\mu_j\}_{j \in \Gamma} \in I^X$.

The pair (X, T) is called a smooth fuzzy topological space.

Remark 1.1. Let (X, T) be a smooth fuzzy topological space. Then, for each $r \in I_0$, $T_r = \{\mu \in I^X : T(\mu) \geq r\}$ is Chang's fuzzy topology on X .

Definition 1.3. [5] Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $C_T : I^X \times I_0 \rightarrow I^X$ is defined as follows: $C_T(\lambda, r) = \wedge\{\mu : \mu \geq \lambda, T(\bar{1} - \mu) \geq r\}$.

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

- (1) $C_T(\bar{0}, r) = \bar{0}$.
- (2) $\lambda \leq C_T(\lambda, r)$.
- (3) $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.
- (4) $C_T(\lambda, r) \leq C_T(\lambda, s)$, if $r \leq s$.
- (5) $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1. [5] Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_T : I^X \times I_0 \rightarrow I^X$ is defined as follows: $I_T(\lambda, r) = \vee\{\mu : \mu \leq \lambda, T(\mu) \geq r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

- (1) $I_T(\bar{1} - \lambda, r) = \bar{1} - C_T(\lambda, r)$.
- (2) $I_T(\bar{1}, r) = \bar{1}$.
- (3) $\lambda \geq I_T(\lambda, r)$.
- (4) $I_T(\lambda, r) \wedge I_T(\mu, r) = I_T(\lambda \wedge \mu, r)$.
- (5) $I_T(\lambda, r) \geq I_T(\lambda, s)$, if $r \leq s$.
- (6) $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

Definition 1.4. [4] Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,

- (1) $SC_T(\lambda, r) = \wedge\{\mu \in I^X : \mu \geq \lambda, \mu \text{ is } r\text{-fuzzy semiclosed}\}$ is called r -fuzzy semiclosure of λ .
- (2) λ is called r -fuzzy semiclosed (briefly, r -fsc) if $\lambda \geq I_T(C_T(\lambda, r), r)$.
- (3) λ is called r -fuzzy semiopen (briefly, r -fso) if $\lambda \leq C_T(I_T(\lambda, r), r)$.

Definition 1.5. [4] Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a function. Then

- (1) f is called fuzzy continuous iff $S(\mu) \leq T(f^{-1}(\mu))$ for each $\mu \in I^Y$.
- (2) f is called fuzzy open iff $T(\lambda) \leq S(f(\lambda))$ for each $\lambda \in I^X$.

§2. Fuzzy $\tilde{g} - T_i$ spaces

In this section, the concept of fuzzy $\tilde{g} - T_i$ ($i = 0, 1, 2, 1/2$) spaces is introduced. Interesting properties and characterizations of such spaces are discussed.

Definition 2.1. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called

- (1) r -fuzzy \hat{g} -closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r -fuzzy semiopen. The complement of a r -fuzzy \hat{g} -closed set is said to be a r -fuzzy \hat{g} -open set.
- (2) r -fuzzy *g -closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r -fuzzy \hat{g} -open. The complement of a r -fuzzy *g -closed set is said to be a r -fuzzy *g -open set.
- (3) r -fuzzy $\#g$ -semiclosed (briefly r - $\#fgs$ -closed) if $SC_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r -fuzzy *g -open. The complement of a r -fuzzy $\#g$ -semiclosed set is said to be a r -fuzzy $\#g$ -semiopen set (briefly r - $\#fgs$ -open set).
- (4) r -fuzzy \tilde{g} -closed if $C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$ and μ is r - $\#fgs$ -open. The complement of a r -fuzzy \tilde{g} -closed set is said to be a r -fuzzy \tilde{g} -open set.

Definition 2.2. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, (1) $\tilde{g} - I_T(\lambda, r) = \vee\{\mu \in I^X : \mu \leq \lambda, \mu \text{ is a } r\text{-fuzzy } \tilde{g}\text{-open set}\}$ is called r -fuzzy \tilde{g} -interior of

λ .

(2) $\tilde{g} - C_T(\lambda, r) = \bigwedge \{ \mu \in I^X : \mu \geq \lambda, \mu \text{ is a } r\text{-fuzzy } \tilde{g}\text{-closed set} \}$ is called r -fuzzy \tilde{g} -closure of λ .

Definition 2.3. Let (X, T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$, λ is called r -generalized fuzzy \tilde{g} -closed (briefly, r -gf \tilde{g} -closed) iff $\tilde{g} - C_T(\lambda, r) \leq \mu$ whenever $\lambda \leq \mu$, $\mu \in I^X$ is r -fuzzy \tilde{g} -open. The complement of a r -generalized fuzzy \tilde{g} -closed set is a r -generalized fuzzy \tilde{g} -open set (briefly, r -gf \tilde{g} -open).

Definition 2.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a function.

(1) f is called \tilde{g} -open (resp. \tilde{g} -closed) if for each r -fuzzy \tilde{g} -open set $\lambda \in I^X$, $f(\lambda) \in I^Y$ is r -fuzzy \tilde{g} -open (resp. r -fuzzy \tilde{g} -closed).

(2) f is called \tilde{g} -continuous if for each $\lambda \in I^Y$ with $S(\lambda) \geq r$, $f^{-1}(\lambda) \in I^X$ is r -fuzzy \tilde{g} -open.

(3) f is called fuzzy \tilde{g} -irresolute if for each r -fuzzy \tilde{g} -open set $\lambda \in I^Y$, $f^{-1}(\lambda) \in I^X$ is r -fuzzy \tilde{g} -open.

(4) f is called fuzzy \tilde{g} -homeomorphism if f is one to one, onto, fuzzy \tilde{g} -irresolute and fuzzy \tilde{g} -open.

(5) f is called gf \tilde{g} -irresolute if for each r -gf \tilde{g} closed set $\lambda \in I^Y$, $f^{-1}(\lambda) \in I^X$ is r -gf \tilde{g} -closed.

(6) f is called gf \tilde{g} -closed iff for any r -gf \tilde{g} -closed set $\lambda \in I^X$, $f(\lambda)$ is r -gf \tilde{g} -closed.

Definition 2.5. A smooth fuzzy topological space (X, T) is called

(1) Fuzzy $\tilde{g} - T_0$ iff for $\lambda, \mu \in I^X$ with $\lambda \bar{q} \mu$, there exists r -fuzzy \tilde{g} -open set $\delta \in I^X$ such that either $\lambda \leq \delta$ or $\mu \leq \delta$, $\lambda \bar{q} \delta$.

(2) Fuzzy $\tilde{g} - T_1$ iff for $\lambda, \mu \in I^X$ with $\lambda \bar{q} \mu$, there exist r -fuzzy \tilde{g} -open sets $\delta, \eta \in I^X$ such that either $\lambda \leq \delta$, $\mu \bar{q} \delta$ or $\mu \leq \eta$, $\lambda \bar{q} \eta$.

(3) Fuzzy $\tilde{g} - T_2$ iff for $\lambda, \mu \in I^X$ with $\lambda \bar{q} \mu$, there exist r -fuzzy \tilde{g} -open sets $\delta, \eta \in I^X$ with $\lambda \leq \delta$, $\mu \leq \eta$ and $\delta \bar{q} \eta$.

(4) Fuzzy $\tilde{g} - R_0$ iff $\lambda \bar{q} \tilde{g} - C_T(\mu, r)$ implies that $\mu \bar{q} \tilde{g} - C_T(\lambda, r)$ for $\lambda, \mu \in I^X$.

Definition 2.6. A smooth fuzzy topological space (X, T) is called fuzzy $\tilde{g} - T_{1/2}$ if every r -gf \tilde{g} -closed set is r -fuzzy \tilde{g} -closed.

Proposition 2.1. Let (X, T) be a smooth fuzzy topological space. For $r \in I_0$, the following properties hold:

(i) For all r -fuzzy \tilde{g} -open set $\lambda \in I^X$, $\lambda q \mu$ iff $\lambda q (\tilde{g} - C_T(\mu, r))$, $\mu \in I^X$.

(ii) $\delta q (\tilde{g} - C_T(\lambda, r))$ iff $\lambda q \mu$ for all r -fuzzy \tilde{g} -open set $\mu \in I^X$ with $\delta \leq \mu$, for $\lambda, \delta \in I^X$.

Proof. (i) Let λ be a r -fuzzy \tilde{g} -open set such that $\lambda q \mu$. Since $\mu \leq \tilde{g} - C_T(\mu, r)$, $\lambda q \tilde{g} - C_T(\mu, r)$. Conversely let λ be a r -fuzzy \tilde{g} -open set such that $\lambda \bar{q} \mu$. Then $\mu \leq \bar{1} - \lambda$, this implies that $\tilde{g} - C_T(\mu, r) \leq \tilde{g} - C_T(\bar{1} - \lambda, r) = \bar{1} - \lambda$. Now, $\tilde{g} - C_T(\mu, r) \leq \bar{1} - \lambda$. Thus $\lambda \bar{q} \tilde{g} - C_T(\mu, r)$ which is a contradiction. Hence the result.

(ii) Let $\delta q (\tilde{g} - C_T(\lambda, r))$. Since $\delta \leq \mu$, $\mu q (\tilde{g} - C_T(\lambda, r))$. By (i), $\mu q \lambda$ for all r -fuzzy \tilde{g} -open set μ with $\delta \leq \mu$. Conversely suppose that $\delta \bar{q} \tilde{g} - C_T(\lambda, r)$. Then $\delta \leq \bar{1} - (\tilde{g} - C_T(\lambda, r))$. Let $\mu = \bar{1} - (\tilde{g} - C_T(\mu, r))$. Then μ is a r -fuzzy \tilde{g} -open set. Since $\lambda \leq \tilde{g} - C_T(\lambda, r)$, $\mu = \bar{1} - (\tilde{g} - C_T(\lambda, r)) \leq \bar{1} - \lambda$, this implies that $\lambda \bar{q} \mu$, a contradiction. Hence the result.

Proposition 2.2. Let (X, T) be a smooth fuzzy topological space. For $\delta, \rho \in I^X$, the following statements are equivalent:

- (i) (X, T) is fuzzy $\tilde{g}-R_0$.
(ii) If $\delta \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$, $\lambda \in I^X$ and $r \in I_0$, there exists a r -fuzzy \tilde{g} -open set μ such that $\delta \bar{q} \mu$ and $\lambda \leq \mu$.
(iii) If $\delta \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$, then $\tilde{g} - C_T(\delta, r) \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$, $\lambda \in I^X$ and $r \in I_0$.
(iv) If $\delta \bar{q} \tilde{g} - C_T(\rho, r)$ then $\tilde{g} - C_T(\delta, r) \bar{q} \tilde{g} - C_T(\rho, r)$, $r \in I_0$.

Proof. (i) \Rightarrow (ii) Let $\delta \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$. Since $\tilde{g} - C_T(\rho, r) \leq \tilde{g} - C_T(\lambda, r)$ for each $\rho \leq \lambda$, we have $\delta \bar{q} (\tilde{g} - C_T(\rho, r))$. By (i), $\rho \bar{q} (\tilde{g} - C_T(\delta, r))$. By (ii) of Proposition 2.1, for each $\rho \bar{q} (\tilde{g} - C_T(\delta, r))$, there exists a r -fuzzy \tilde{g} -open set η such that $\delta \bar{q} \eta$, $\rho \leq \eta$. Let $\mu = \bigvee \{ \eta : \delta \bar{q} \eta \}$. Then $\delta \bar{q} \mu$, $\lambda \leq \mu$ for all r -fuzzy \tilde{g} -open set μ .

(ii) \Rightarrow (iii) Let $\delta \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$. By (ii), there exists a r -fuzzy \tilde{g} -open set $\mu \in I^X$ such that $\delta \bar{q} \mu$ and $\lambda \leq \mu$. Since $\delta \bar{q} \mu$, it follows that $\delta \leq \bar{1} - \mu$. This implies that $\tilde{g} - C_T(\delta, r) \leq \tilde{g} - C_T(\bar{1} - \mu, r) = \bar{1} - \mu \leq \bar{1} - \lambda$. Hence $\tilde{g} - C_T(\delta, r) \bar{q} \lambda = \tilde{g} - C_T(\lambda, r)$.

(iii) \Rightarrow (iv) Let $\delta \bar{q} \tilde{g} - C_T(\rho, r)$. Since $\tilde{g} - C_T(\tilde{g} - C_T(\rho, r), r) = \tilde{g} - C_T(\rho, r)$ and by (iii), $\tilde{g} - C_T(\delta, r) \bar{q} \tilde{g} - C_T(\rho, r)$.

(iv) \Rightarrow (i) Let $\delta \bar{q} \tilde{g} - C_T(\rho, r)$. By (iv), $\tilde{g} - C_T(\delta, r) \bar{q} \tilde{g} - C_T(\rho, r)$. Since $\rho \leq \tilde{g} - C_T(\rho, r)$, $\rho \bar{q} \tilde{g} - C_T(\delta, r)$. Hence (X, T) is fuzzy $\tilde{g}-R_0$.

Proposition 2.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy \tilde{g} -irresolute, $gf\tilde{g}$ -irresolute and fuzzy \tilde{g} -closed function. Then the following conditions hold:

- (i) If f is injective and (Y, S) is a fuzzy $\tilde{g}-T_{1/2}$ space, then (X, T) is a fuzzy $\tilde{g}-T_{1/2}$ space.
(ii) If f is surjective and (X, T) is a fuzzy $\tilde{g}-T_{1/2}$ space, then (Y, S) is a fuzzy $\tilde{g}-T_{1/2}$ space.

Proof. (i) Let $\lambda \in I^X$ be a r - $gf\tilde{g}$ -closed set. Since f is $gf\tilde{g}$ -closed, $f(\lambda) \in I^Y$ is r - $gf\tilde{g}$ -closed. Since (Y, S) is fuzzy $\tilde{g}-T_{1/2}$, $f(\lambda)$ is r -fuzzy \tilde{g} -closed. Now, $\lambda = f^{-1}(f(\lambda))$ is r -fuzzy \tilde{g} -closed. Hence (X, T) is a fuzzy $\tilde{g}-T_{1/2}$ space.

(ii) Let $\mu \in I^Y$ be a r - $gf\tilde{g}$ -closed set. Since f is $gf\tilde{g}$ -irresolute, $f^{-1}(\mu) \in I^X$ is a r - $gf\tilde{g}$ -closed set. Since (X, T) is a fuzzy $\tilde{g}-T_{1/2}$ space, $f^{-1}(\mu)$ is a r -fuzzy \tilde{g} -closed set. Therefore $\mu = f(f^{-1}(\mu))$ is r -fuzzy \tilde{g} -closed. Hence (Y, S) is a fuzzy $\tilde{g}-T_{1/2}$ space.

Proposition 2.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. Let $f : (X, T) \rightarrow (Y, S)$ be a fuzzy \tilde{g} -irresolute, and injective function. If (Y, S) is fuzzy $\tilde{g}-T_2$ (resp. fuzzy $\tilde{g}-T_1$), then (X, T) is fuzzy $\tilde{g}-T_2$ (resp. fuzzy $\tilde{g}-T_1$).

Proof. Let (Y, S) be a fuzzy $\tilde{g}-T_2$ space. Let $\lambda_1, \lambda_2 \in I^X$ be such that $\lambda_1 \bar{q} \lambda_2$, then exist r -fuzzy \tilde{g} -open sets $\lambda, \mu \in I^Y$ such that $f(\lambda_1) \leq \lambda$ and $f(\lambda_2) \leq \mu$ such that $\lambda \bar{q} \mu$. Then $\lambda \leq \bar{1} - \mu$ which implies that $f^{-1}(\lambda) \bar{q} f^{-1}(\mu)$. Now, $\lambda_1 \leq f^{-1}(\lambda)$ and $\lambda_2 \leq f^{-1}(\mu)$. Since f is fuzzy \tilde{g} -irresolute, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are r -fuzzy \tilde{g} -open sets. Hence (X, T) is a fuzzy $\tilde{g}-T_2$ space. Similarly we prove the case of fuzzy $\tilde{g}-T_1$ space.

§3. Fuzzy \tilde{g} -normal spaces and its characterizations

In this section, the concept of fuzzy \tilde{g} -normal space is introduced. Interesting properties and characterizations of such space are discussed.

Definition 3.1. A smooth fuzzy topological space (X, T) is said to be fuzzy \tilde{g} -normal if for every r -fuzzy \tilde{g} -closed set λ and r -fuzzy \tilde{g} -open set μ with $\lambda \leq \mu$ there exists a $\gamma \in I^X$ such

that $\lambda \leq \tilde{g} - I_T(\gamma, r) \leq \tilde{g} - C_T(\gamma, r) \leq \mu, r \in I_0$.

Proposition 3.1. For any smooth fuzzy topological space (X, T) and $\lambda, \mu, \delta \in I^X, r \in I_0$, the following statements are equivalent:

- (i) (X, T) is fuzzy \tilde{g} -normal.
- (ii) For each r-fuzzy \tilde{g} -closed set λ and each r-fuzzy \tilde{g} -open set μ with $\lambda \leq \mu$, there exists a r-fuzzy \tilde{g} -open set δ such that $\tilde{g} - C_T(\lambda, r) \leq \delta \leq \tilde{g} - C_T(\delta, r) \leq \mu$.
- (iii) For each r-gf \tilde{g} -closed set λ and r-fuzzy \tilde{g} -open set μ with $\lambda \leq \mu$, there exists a r-fuzzy \tilde{g} -open set δ such that $\tilde{g} - C_T(\lambda, r) \leq \delta \leq \tilde{g} - C_T(\delta, r) \leq \mu$.

Proof. (i) \Rightarrow (ii) The proof is trivial.

(ii) \Rightarrow (iii) Let λ be any r-gf \tilde{g} -closed set and μ be any r-fuzzy \tilde{g} -open set such that $\lambda \leq \mu$. Since λ is r-gf \tilde{g} -closed, $\tilde{g} - C_T(\lambda, r) \leq \mu$. Now, $\tilde{g} - C_T(\lambda, r)$ is r-fuzzy \tilde{g} -closed and μ is r-fuzzy \tilde{g} -open. By (ii), there exists a r-fuzzy \tilde{g} -open set δ such that $\tilde{g} - C_T(\lambda, r) \leq \delta \leq \tilde{g} - C_T(\delta, r) \leq \mu$.

(iii) \Rightarrow (i) The proof is trivial.

Proposition 3.2. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is fuzzy \tilde{g} -homeomorphism and (Y, S) is fuzzy \tilde{g} -normal, then (X, T) is fuzzy \tilde{g} -normal.

Proof. Let $\lambda \in I^X$ be any r-fuzzy \tilde{g} -closed set and $\mu \in I^X$ be any r-fuzzy \tilde{g} -open set such that $\lambda \leq \mu$ where $r \in I_0$. Since f is fuzzy \tilde{g} -homeomorphism, it is also fuzzy \tilde{g} -closed. Hence $f(\lambda) \in I^Y$ is r-fuzzy \tilde{g} -closed. Since f is fuzzy \tilde{g} -open, $f(\mu) \in I^Y$ is r-fuzzy \tilde{g} -open. Since (Y, S) is fuzzy \tilde{g} -normal, there exists a $\gamma \in I^Y$ such that $f(\lambda) \leq \tilde{g} - I_T(\gamma, r) \leq \tilde{g} - C_T(\gamma, r) \leq f(\mu)$. Now, $f^{-1}(f(\lambda)) = \lambda \leq f^{-1}(\tilde{g} - I_T(\gamma, r)) \leq f^{-1}(\tilde{g} - C_T(\gamma, r)) \leq f^{-1}(f(\mu)) = \mu$. That is, $\lambda \leq \tilde{g} - I_T(f^{-1}(\gamma), r) \leq \tilde{g} - C_T(f^{-1}(\gamma), r) \leq \mu$. Therefore (X, T) is fuzzy \tilde{g} -normal.

Proposition 3.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is fuzzy \tilde{g} -homeomorphism and (X, T) is a fuzzy \tilde{g} -normal space, then (Y, S) is fuzzy \tilde{g} -normal.

Proof. Let $\lambda \in I^Y$ be any r-fuzzy \tilde{g} -closed set and $\mu \in I^Y$ be any r-fuzzy \tilde{g} -open set such that $\lambda \leq \mu$ where $r \in I_0$. Since f is fuzzy \tilde{g} -irresolute, $f^{-1}(\lambda)$ is r-fuzzy \tilde{g} -closed and $f^{-1}(\mu) \in I^X$ is r-fuzzy \tilde{g} -open. Since (X, T) is fuzzy \tilde{g} -normal, there exists a $\gamma \in I^X$ such that $f^{-1}(\lambda) \leq \tilde{g} - I_T(\gamma, r) \leq \tilde{g} - C_T(\gamma, r) \leq f^{-1}(\mu)$. Now, $f(f^{-1}(\lambda)) = \lambda \leq f(\tilde{g} - I_T(\gamma, r)) \leq f(\tilde{g} - C_T(\gamma, r)) \leq f(f^{-1}(\mu)) = \mu$. That is, $\lambda \leq \tilde{g} - I_T(f(\gamma), r) \leq \tilde{g} - C_T(f(\gamma), r) \leq \mu$. Therefore (Y, S) is fuzzy \tilde{g} -normal.

Proposition 3.4. Let (X, T) be a smooth fuzzy topological space which is also a fuzzy \tilde{g} -normal space. Let $\{\lambda_i\}_{i \in J} \in I^X$ and $\{\mu_j\}_{j \in J} \in I^X$. If there exist $\lambda, \mu \in I^X$ such that $\tilde{g} - C_T(\lambda_i, r) \leq \tilde{g} - C_T(\lambda, r) \leq \tilde{g} - I_T(\mu_j, r)$ and $\tilde{g} - C_T(\lambda_i, r) \leq \tilde{g} - I_T(\mu, r) \leq \tilde{g} - I_T(\mu_j, r)$ for all $i, j = 1, 2, \dots$, and $r \in I_0$, then there exists $\gamma \in I^X$ such that

$$\tilde{g} - C_T(\lambda_i, r) \leq \tilde{g} - I_T(\gamma, r) \leq \tilde{g} - C_T(\gamma, r) \leq \tilde{g} - I_T(\mu_j, r), \text{ for all } i, j = 1, 2, \dots$$

Proof. First, we shall show by induction that for all $n \geq 2$ there exists a collection $\{\gamma_i, \delta_i/1 \leq i \leq n\}$ contained in I^X such that the conditions

$$\left. \begin{aligned} \tilde{g} - C_T(\lambda_i, r) &\leq \tilde{g} - I_T(\gamma_i, r); \\ \tilde{g} - C_T(\delta_j, r) &\leq \tilde{g} - I_T(\mu_j, r); \\ \tilde{g} - C_T(\lambda, r) &\leq \tilde{g} - I_T(\delta_j, r); \\ \tilde{g} - C_T(\gamma_i, r) &\leq \tilde{g} - I_T(\mu, r); \\ \tilde{g} - C_T(\gamma_i, r) &\leq \tilde{g} - I_T(\delta_j, r), \end{aligned} \right\} (s_n)$$

hold for all $i, j = 1, 2, \dots, n-1$. Clearly (S_2) follows at once from the fuzzy \tilde{g} -normality of (X, T) . Now, suppose that for $n \geq 2$, $\gamma_i, \delta_i \in I^X (i < n)$ such that (S_n) holds. Since $\tilde{g} - C_T(\lambda_n, r) \leq \tilde{g} - C_T(\lambda, r) \leq \tilde{g} - I_T(\delta_j, r) (j < n)$ and $\tilde{g} - C_T(\lambda_n, r) \leq \tilde{g} - I_T(\mu, r)$ by fuzzy \tilde{g} -normality of (X, T) , there exists $\gamma_n \in I^X$ such that

$$\tilde{g} - C_T(\lambda_n, r) \leq \tilde{g} - I_T(\gamma_n, r) \leq \tilde{g} - C_T(\gamma_n, r) \leq \tilde{g} - I_T \left(\bigwedge_{j < n} \delta_j \wedge \mu, r \right).$$

Similarly, since $\tilde{g} - C_T(\lambda, r) \leq \tilde{g} - I_T(\mu_n, r)$ and $\tilde{g} - C_T(\gamma_i, r) \leq \tilde{g} - I_T(\mu_n, r) (i \leq n)$, there exists $\delta_n \in I^X$ such that

$$\begin{aligned} \left(\bigvee_{i \leq n} \tilde{g} - C_T(\gamma_i, r) \vee \tilde{g} - C_T(\lambda, r) \right) &\leq \tilde{g} - I_T(\delta_n, r) \\ &\leq \tilde{g} - C_T(\delta_n, r) \\ &\leq \tilde{g} - I_T(\mu_n, r). \end{aligned}$$

Thus (S_{n+1}) holds.

Let $\gamma = \bigvee_{i=1}^{\infty} \gamma_i$. Then $\tilde{g} - C_T(\lambda_i, r) \leq \tilde{g} - I_T(\gamma_i, r) \leq \tilde{g} - I_T(\gamma, r)$ for all $i = 1, 2, \dots$. Since $\tilde{g} - C_T(\gamma_i, r) \leq \tilde{g} - I_T(\delta_j, r) (i, j = 1, 2, \dots)$, $\gamma_i \leq \delta_j$, so that $\tilde{g} - C_T(\gamma, r) \leq \tilde{g} - C_T(\delta_j, r) \leq \tilde{g} - I_T(\mu_j, r)$ for all $j = 1, 2, \dots$. This proves the result.

Proposition 3.5. Let (X, T) be a smooth fuzzy topological space which is also a fuzzy \tilde{g} -normal space. If $\{\lambda_q\}_{q \in Q}$ and $\{\mu_q\}_{q \in Q}$ are monotone increasing collections of respectively, fuzzy \tilde{g} -closed and fuzzy \tilde{g} -open subsets of (X, T) (Q is the set of all rational numbers) such that $\lambda_q \leq \mu_s$ whenever $q < s$, then there exists a collection $\{\gamma_q\}_{q \in Q} \in I^X$ such that

$$\lambda_q \leq \tilde{g} - I_T(\gamma_s, r), \tilde{g} - C_T(\gamma_q, r) \leq \tilde{g} - I_T(\gamma_s, r) \text{ and } \tilde{g} - C_T(\gamma_q, r) \leq \mu_s$$

whenever $q < s$.

Proof. Let us arrange into a sequence q_n of all rational numbers (without repetitions). For every $n \geq 2$ we shall define inductively a collection $\{\gamma_{q_i} / 1 \leq i \leq n\} \in I^X$ such that

$$\left. \begin{aligned} \lambda_q &\leq \tilde{g} - I_T(\gamma_{q_i}, r), & \text{if } q < q_i; \\ \tilde{g} - C_T(\gamma_{q_i}, r) &\leq \mu_q, & \text{if } q_i < q; \\ \tilde{g} - C_T(\gamma_{q_i}, r) &\leq \tilde{g} - I_T(\gamma_{q_j}, r), & \text{if } q_i < q_j, \end{aligned} \right\} (s_n)$$

for all $1 \leq i, j < n$. It is clear that the countable collections $\{\lambda_q / q < q_1\}$ and $\{\mu_q / q > q_1\}$ together with λ_{q_1} and μ_{q_1} satisfy all hypotheses of Proposition 3.4, so that there exists $\delta_1 \in I^X$ such that $\lambda_q \leq \tilde{g} - I_T(\delta_1, r)$ for all $q < q_1$ and $\tilde{g} - C_T(\delta_1, r) \leq \mu_q$ for all $q > q_1$.

Letting $\gamma_{q_1} = \delta_1$, we get (S_2) . Assume that the fuzzy subsets γ_{q_i} are already defined for $i < n$ and satisfy (S_n) . Define

$$\lambda = \vee \{ \gamma_{q_i} / i < n, q_i < q_n \} \vee \lambda_{q_n} \text{ and } \mu = \wedge \{ \gamma_{q_j} / j < n, q_j < q_n \} \wedge \mu_{q_n}.$$

Then $\tilde{g} - C_T(\gamma_{q_i}, r) \leq \tilde{g} - C_T(\lambda, r) \leq \tilde{g} - I_T(\gamma_{q_j}, r)$, and $\tilde{g} - C_T(\gamma_{q_i}, r) \leq \tilde{g} - I_T(\mu, r) \leq \tilde{g} - I_T(\gamma_{q_j}, r)$ whenever $q_i < q_n < q_j$, $(i, j < n)$ as well as $\lambda_q \leq \tilde{g} - C_T(\lambda, r) \leq \mu_s$ and $\lambda_q \leq \tilde{g} - I_T(\mu, r) \leq \mu_s$ whenever $q < q_n < s$. This shows that the countable collections $\{ \gamma_{q_i} / i < n, q_i < q_n \} \vee \{ \lambda_q / q < q_n \}$ and $\{ \gamma_{q_j} / j < n, q_j > q_n \} \vee \{ \mu_q / q > q_n \}$ together with λ and μ satisfy all hypotheses of Proposition 3.5. Hence there exists a $\delta_n \in I^X$ such that

$$\begin{aligned} \lambda_q &\leq \tilde{g} - I_T(\delta_n, r), & \text{if } q < q_n \\ \tilde{g} - C_T(\gamma_{q_i}, r) &\leq \tilde{g} - I_T(\delta_n, r), & \text{if } q_i < q_n \\ \tilde{g} - C_T(\delta_n, r) &\leq \mu_q, & \text{if } q_n < q \\ \tilde{g} - C_T(\delta_n, r) &\leq \tilde{g} - I_T(\gamma_{q_j}, r), & \text{if } q_n < q_j \end{aligned}$$

where $1 \leq i, j \leq n-1$. Letting $\gamma_{q_n} = \delta_n$ we obtain fuzzy subsets $\gamma_{q_1}, \gamma_{q_2}, \dots, \gamma_{q_n}$ that satisfy the result (S_{n+1}) . Therefore the collection $\{ \gamma_{q_i} / i = 1, 2, \dots \}$ has the required properties. This completes the proof.

§4. Fuzzy \tilde{g} -regular space and its characterizations

In this section, the concept of fuzzy \tilde{g} -regular space is introduced. Some interesting characterizations are established.

Definition 4.1. A smooth fuzzy topological space (X, T) is called a fuzzy \tilde{g} -regular space if for every r-fuzzy \tilde{g} -closed set λ and each $\alpha \in I^X$ with $\alpha \not\leq \lambda$, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta \bar{q} \mu$ such that $\alpha \leq \delta, \lambda \leq \mu$.

Proposition 4.1. Let (X, T) be a smooth fuzzy topological space. Then the following statements are equivalent:

- (i) (X, T) is fuzzy \tilde{g} -regular.
- (ii) For each $\alpha \in I^X$ and r-fuzzy \tilde{g} -open set λ with $\alpha \bar{q} \lambda$ there exists a $\delta \in I^X$ with $T(\delta) \geq r, \alpha \leq \delta$ such that $C_T(\delta, r) \leq \lambda$.

Proof. (i) \Rightarrow (ii) Let λ be any r-fuzzy \tilde{g} -open set with $\alpha \bar{q} \lambda$. By hypothesis, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta \bar{q} \mu$ such that $\bar{1} - \lambda \leq \mu$ and $\alpha \leq \delta$. Since $\delta \leq \bar{1} - \mu, C_T(\delta, r) \leq C_T(\bar{1} - \mu, r) = \bar{1} - \mu$. But $\bar{1} - \lambda \leq \mu$ gives $\bar{1} - \mu \leq \lambda$. That is, $C_T(\delta, r) \leq \lambda$. Hence the result.

(ii) \Rightarrow (i) Let γ be any r-fuzzy \tilde{g} -closed set with $\alpha \not\leq \gamma$ for any $\alpha \in I^X$. Now, $\bar{1} - \gamma$ is r-fuzzy \tilde{g} -open. By hypothesis, there exists a $\delta \in I^X$ with $T(\delta) \geq r, \alpha \leq \delta$ such that $C_T(\delta, r) \leq \bar{1} - \gamma$. Then $\gamma \leq \bar{1} - C_T(\delta, r)$. Now, $\delta \leq \bar{1} - (\bar{1} - C_T(\delta, r))$ such that $\alpha \leq \delta$ and $\gamma \leq \bar{1} - C_T(\delta, r)$. Therefore (X, T) is fuzzy \tilde{g} -regular.

Proposition 4.2. Let (X, T) be a smooth fuzzy topological space. Then (X, T) is fuzzy \tilde{g} -regular iff for every r-fuzzy \tilde{g} -closed set $\lambda \in I^X$ and $\alpha \in I^X$ with $\alpha \not\leq \lambda$, there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ such that $\alpha \leq \delta, \lambda \leq \mu$, then $\mu \bar{q} C_T(\delta, r)$ where $r \in I_0$.

Proof. Let (X, T) be a fuzzy \tilde{g} -regular space. Let λ be any r-fuzzy \tilde{g} -closed set and α be such that $\alpha \not\leq \lambda$. Since (X, T) is fuzzy \tilde{g} -regular, there exist μ, δ with $T(\mu) \geq r, T(\delta) \geq r, \delta \bar{q} \mu$ such that $\alpha \leq \delta, \lambda \leq \mu$. Now, $\delta \bar{q} \mu$ implies that $C_T(\delta, r) \leq C_T(\bar{1} - \mu, r) = \bar{1} - \mu$. That is, $\mu \bar{q} C_T(\delta, r)$. Hence the result. Converse part is trivial.

Proposition 4.3. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is bijective, fuzzy \tilde{g} -irresolute, fuzzy open and if (X, T) is a fuzzy \tilde{g} -regular space, then (Y, S) is fuzzy \tilde{g} -regular.

Proof. Let $\lambda \in I^Y$ be any r-fuzzy \tilde{g} -closed set and $\beta \in I^Y$ be such that $\beta \not\leq \lambda, r \in I_0$. Since f is fuzzy \tilde{g} -irresolute, $f^{-1}(\lambda) \in I^X$ is r-fuzzy \tilde{g} -closed. Let $f(\alpha) = \beta$ for any $\alpha \in I^X$. Since f is bijective, $\alpha = f^{-1}(\beta)$. Since (X, T) is fuzzy \tilde{g} -regular and $\alpha \not\leq f^{-1}(\lambda)$ there exist $\mu, \delta \in I^X$ with $T(\mu) \geq r, T(\delta) \geq r$ and $\delta \bar{q} \mu$ such that $\alpha \leq \delta$ and $f^{-1}(\lambda) \leq \mu$. Since f is fuzzy open and bijective, $f(\alpha) \leq f(\delta)$ implies that $\beta \leq f(\delta), \lambda \leq f(\mu)$ and $S(f(\delta)) \geq r, S(f(\mu)) \geq r$ with $f(\delta) \bar{q} f(\mu)$. Hence (Y, S) is fuzzy \tilde{g} -regular.

Proposition 4.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces. If $f : (X, T) \rightarrow (Y, S)$ is fuzzy \tilde{g} -closed, fuzzy continuous, injective and (Y, S) is fuzzy \tilde{g} -regular then (X, T) is fuzzy \tilde{g} -regular.

Proof. Let $\lambda \in I^X$ be any r-fuzzy \tilde{g} -closed set and $\alpha \in I^X$ be such that $\alpha \not\leq \lambda, r \in I_0$. Since f is fuzzy \tilde{g} -closed, $f(\lambda) \in I^Y$ is r-fuzzy \tilde{g} -closed and $f(\alpha) \not\leq f(\lambda)$. Since (Y, S) is fuzzy \tilde{g} -regular, there exist $\mu, \delta \in I^Y$ with $S(\mu) \geq r, S(\delta) \geq r$ and $\delta \bar{q} \mu$ such that $f(\alpha) \leq \mu$ and $f(\lambda) \leq \delta$. Since f is fuzzy continuous, $f^{-1}(\mu), f^{-1}(\delta) \in I^X$ with $T(f^{-1}(\mu)) \geq r$ and $T(f^{-1}(\delta)) \geq r$. Also, $\alpha \leq f^{-1}(\mu), \lambda \leq f^{-1}(\delta)$ and $f^{-1}(\delta) \bar{q} f^{-1}(\mu)$. Therefore (X, T) is fuzzy \tilde{g} -regular.

Proposition 4.5. Let (X, T) be a smooth fuzzy topological space. Then the following statements are equivalent:

- (i) (X, T) is fuzzy \tilde{g} -regular.
- (ii) For every r-fuzzy \tilde{g} -open set λ such that $\alpha \leq \lambda$ there exists a $\gamma \in I^X$ with $T(\gamma) \geq r$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$.
- (iii) For every r-fuzzy \tilde{g} -open set λ such that $\alpha \leq \lambda$ there exists a $\delta \in I^X$ with $T(\delta) \geq r$ and $\delta = I_T(\Delta, r), T(\bar{1} - \Delta) \geq r$ such that $\alpha \leq \delta \leq C_T(\delta, r) \leq \lambda$.
- (iv) For every r-fuzzy \tilde{g} -closed set μ such that $\alpha \not\leq \mu$ there exist γ and λ with $T(\gamma) \geq r$ and $T(\lambda) \geq r$ such that $\alpha \leq \gamma, \mu \leq \lambda$ with $C_T(\gamma, r) \bar{q} C_T(\lambda, r)$.

Proof. (i) \Rightarrow (ii) Let λ be a r-fuzzy \tilde{g} -open set such that $\alpha \leq \lambda$. Then $\bar{1} - \lambda$ is a r-fuzzy \tilde{g} -closed set such that $\alpha \not\leq \bar{1} - \lambda$. Since (X, T) is \tilde{g} -regular, there exist $\gamma, \delta \in I^X$ with $T(\gamma) \geq r, T(\delta) \geq r$ and $\gamma \bar{q} \delta$ such that $\alpha \leq \gamma, \bar{1} - \lambda \leq \delta$. Since $\gamma \bar{q} \delta, \gamma \leq \bar{1} - \delta$. Hence $C_T(\gamma, r) \leq C_T(\bar{1} - \delta, r) = \bar{1} - \delta$. But $\bar{1} - \delta \leq \lambda$. Therefore $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$.

(ii) \Rightarrow (iii) Let λ be a r-fuzzy \tilde{g} -open set such that $\alpha \leq \lambda$. By (ii), there exists a $\gamma \in I^X$ with $T(\gamma) \geq r$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \lambda$. Let $\delta = I_T(\Delta, r)$ where $\Delta = C_T(\gamma, r)$. Now, $\alpha \leq \gamma \leq I_T(\Delta, r) \leq C_T(\Delta, r) \leq \lambda$. Also, then $\alpha \leq \delta \leq C_T(\delta, r) = C_T(I_T(\Delta, r), r) \leq C_T(\Delta, r) = C_T(C_T(\gamma, r), r) = C_T(\gamma, r) \leq \lambda$. Thus $\alpha \leq \delta \leq C_T(\delta, r) \leq \lambda$.

(iii) \Rightarrow (iv) Let μ be a r-fuzzy \tilde{g} -closed set with $\alpha \not\leq \mu$. Then $\bar{1} - \mu$ is a r-fuzzy \tilde{g} -open set with $\alpha \leq \bar{1} - \mu$. By (iii), there exists a $\delta \in I^X$ with $T(\delta) \geq r$ such that $\alpha \leq \delta \leq C_T(\delta, r) \leq \bar{1} - \mu$ where $\delta = I_T(\Delta, r)$ for some $\Delta \in I^X$ with $T(\bar{1} - \Delta) \geq r$. Again by hypothesis there exists a γ

$\in I^X$ such that $\alpha \leq \gamma \leq C_T(\gamma, r) \leq \delta$. Let $\lambda = \bar{1} - C_T(\delta, r)$. Then $\alpha \leq \gamma$, $\mu \leq \lambda$ with $\lambda \leq \bar{1} - \delta$. Now, $C_T(\lambda, r) \leq \bar{1} - \delta \leq \bar{1} - C_T(\gamma, r)$. Thus $C_T(\gamma, r) \bar{q} C_T(\lambda, r)$.

(iv) \Rightarrow (i) The proof is trivial.

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An integral identity involving the Hermite polynomials¹

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Abstract The main purpose of this paper is using the elementary method and the properties of the power series to study an integral calculating problem involving the Hermite polynomials, then give an interesting identity.

Keywords Hermite polynomials, integral calculation, identity, elementary method.

§1. Introduction

For any real number x , the polynomial solutions

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \sum_{0 \leq k \leq \frac{n}{2}} \frac{(-1)^k n!}{k!(n-2k)!} (2x)^{n-2k}$$

of the Hermite equations

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0 \quad (n = 0, 1, 2, \dots)$$

are called Hermite polynomials, see [1]. For example, the first several polynomials are: $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$, $H_5(x) = 32x^5 - 160x^3 + 120x$, \dots . It is well know that $H_n(x)$ is an orthogonality polynomial. That is,

$$\int_{-\infty}^{+\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0, & \text{if } m \neq n; \\ 2^n n! \sqrt{\pi}, & \text{if } m = n. \end{cases}$$

And it play a very important rule in the theories and applications of mathematics. So there are many people had studied its properties, some results and related papers see references [2], [3], [4], [5] and [6].

In this paper, we shall study the calculating problem of the integral

$$\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} H_{a_1}(x) H_{a_2}(x) \cdots H_{a_k}(x) dx, \quad (1)$$

and give an interesting calculating formula for it. About this problem, it seems that no one had studied yet, at least we have not seen any related papers before. The problem is interesting,

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because it can help us to know more information about the orthogonality of $H_n(x)$. The main purpose of this paper is using the elementary method and the properties of the power series to give an exact calculating formula for (1). That is, we shall prove the following conclusions:

Theorem 1. Let n and k are two positive integer with $n \geq k$, then we have the identity

$$\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \dots \frac{H_{a_k}(x)}{a_k!} dx = \begin{cases} 0, & \text{if } n = 2m - 1; \\ \frac{\sqrt{\pi}(k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases}$$

where $\sum_{a_1+a_2+\dots+a_k=n}$ denotes the summation over all nonnegative integers a_1, a_2, \dots, a_k such that $a_1 + a_2 + \dots + a_k = n$.

Theorem 2. Let m, n and k are positive integers with $n \geq k \geq 1$, then we have the identity

$$\begin{aligned} \sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1+m}^{(m)}(x)}{(a_1+m)!} \frac{H_{a_2+m}^{(m)}(x)}{(a_2+m)!} \dots \frac{H_{a_k+m}^{(m)}(x)}{(a_k+m)!} dx \\ = \begin{cases} 0, & \text{if } n = 2m - 1; \\ \frac{\sqrt{\pi} \cdot 2^{km} \cdot (k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases} \end{aligned}$$

where $H_n^{(m)}(x)$ denotes the m -th derivative of $H_n(x)$ for x .

From Theorem 1 and Theorem 2 we know that the integration must be 0, if n be an odd number. So it is interesting that the orthogonality in such an integral only depend on the parity of n .

§2. Proof of the theorems

In this section, we shall use the elementary method and the properties of the power series to prove our Theorems directly. First we prove Theorem 1. For any positive integer k , from the generating function of $H_n(x)$ and the properties of the power series we have

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

and

$$e^{k2xt-kt^2} = \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\dots+a_k=n} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \dots \frac{H_{a_k}(x)}{a_k!} \right) t^n. \quad (2)$$

So from (2) we may get

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{2kxt-kt^2-x^2} dx &= \int_{-\infty}^{+\infty} e^{-(x-kt)^2+(k^2-k)t^2} dx \\ &= \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \dots \frac{H_{a_k}(x)}{a_k!} dx \right) t^n. \end{aligned} \quad (3)$$

On the other hand, for any real number t and integer x , note that the integral

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{-(x-kt)^2+(k^2-k)t^2} dx &= e^{(k^2-k)t^2} \int_{-\infty}^{+\infty} e^{-(x-kt)^2} dx \\ &= e^{(k^2-k)t^2} \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} e^{(k^2-k)t^2} \end{aligned} \quad (4)$$

and

$$e^{(k^2-k)t^2} = \sum_{n=0}^{\infty} \frac{(k^2-k)^n}{n!} t^{2n}. \quad (5)$$

Combining (2), (3), (4) and (5) we may get

$$\begin{aligned} &\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(k^2-k)^n}{n!} t^{2n} \\ &= \sum_{n=0}^{\infty} \left(\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \dots \frac{H_{a_k}(x)}{a_k!} dx \right) t^n. \end{aligned} \quad (6)$$

Comparing the coefficients of t^n in (6) we may immediately deduce the identity

$$\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1}(x)}{a_1!} \frac{H_{a_2}(x)}{a_2!} \dots \frac{H_{a_k}(x)}{a_k!} dx = \begin{cases} 0, & \text{if } n = 2m-1; \\ \frac{\sqrt{\pi}(k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases}$$

where $\sum_{a_1+a_2+\dots+a_k=n}$ denotes the summation over all nonnegative integers a_1, a_2, \dots, a_k such that $a_1 + a_2 + \dots + a_k = n$. This proves Theorem 1.

Now we prove Theorem 2. Note that

$$\frac{d^m}{dx^m} e^{2xt-t^2} = (2t)^m e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_{n+m}^{(m)}(x)}{(n+m)!} t^{n+m}$$

or

$$2^m e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_{n+m}^{(m)}(x)}{(n+m)!} t^n.$$

Then using the method of proving Theorem 1 we may get

$$\begin{aligned} &\sum_{a_1+a_2+\dots+a_k=n} \int_{-\infty}^{+\infty} e^{-x^2} \frac{H_{a_1+m}^{(m)}(x)}{(a_1+m)!} \frac{H_{a_2+m}^{(m)}(x)}{(a_2+m)!} \dots \frac{H_{a_k+m}^{(m)}(x)}{(a_k+m)!} dx \\ &= \begin{cases} 0, & \text{if } n = 2m-1; \\ \frac{\sqrt{\pi} \cdot 2^{km} \cdot (k^2-k)^m}{m!}, & \text{if } n = 2m. \end{cases} \end{aligned}$$

where $H_n^{(m)}(x)$ denotes the m -th derivative of $H_n(x)$ for x .

This completes the proof of Theorem 2.

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Vinegar identification by ultraviolet spectrum technology and pattern recognition method

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Abstract Some kinds of vinegars are studied for identification. First, their ultraviolet spectrum curves are obtained by evaporation and ultraviolet spectrum scanning under the conditions of wavelength at 245~330nm, quantity dilution ratio of the liquid at 1:6, evaporation temperature at 45°C, mass concentration of reference at 45g/L. Then, the data are analyzed by the method of pattern recognition, such as Euclid (Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. The identification accuracy of Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP are 100%. The results show that these methods can be effective ways to identify vinegar.

Keywords Ultraviolet Spectrum, Vinegar, identification, Hybrid Discriminant Analysis, BP Neural Network.

§1. Introduction

There are many varieties of vinegar in the market nowadays, but their qualities are not same, and there are not quick and valid methods to identify them. For a long period, people identify them from some sense index such as color, smelling, taste, style and some simple quantity index by experience; these methods are of subjectivity and unilateralism by all appearances.

Zhang Shunping et. al. measured vinegar by electronic nose technology, analyzing the comparability and similitude degree of vinegar at the aspect of savour, class and acidity by clustering and principal component analysis method, he also recognised the vinegar by probability neural network, the accuracy of recognition is 94.4% [1].

Ultraviolet Spectrum technology is a new measure technique, since there are different components in different matter systems and the unsaturation degree of the components are not same, the UV abs curves of the matter system are different. We can identify the matter systems by comparing their UV abs curves. In our previous works [2], we took the similitude degree of the curve as index, test the recurring, stability and otherness of the ultraviolet spectrum method,

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we concluded that this method can identify the vinegars. In this paper, we further study the method of vinegar identification, we process and analyzed the data of UV abs curves by pattern recognition method, such as Euclid(Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. We recognize 5 kinds of vinegar samples, the accuracy of recognition is 100% when using Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP.

§2. The material and data

2.1. Experiment material

Vinegars: (a)Black Rice spicy vinegar, (b) Jiajia mature vinegar, (c)Shuita mature vinegar, (d)Xiaoerhei grain spicy vinegar, (e)Zhenjiang spicy vinegar, are all bought from Yangling Guomao Supermarket.

Self-made vinegar: the samples are from the Practice Factory of College of Food Science and Engineering, Northwest A and F University.

Reagent: Glacial acetic acid, sodium hydroxide, are all analysis pure reagent and made in China.

Water for experiment: distilled water.

2.2. Instrument for experiment:

BUCHI Rotavapor R-200 circumrotate evaporation instrument(BUCHI Company), UV-2550 double beam of ultraviolet—visible light photometer(Japan).

2.3. Condition of experiment and data

With the scanning wavelength range at 245~330nm, sampling interval at 0.5nm, aperture width at 0.5nm, dilution ratio of evaporated liquor at 1:6, evaporation temperature at 45°C, quality thickness of reference fluid (glacial acetic acid) at 45g/L, we scan the vinegars with ultraviolet spectrum at different storage time, and obtain the data, see [2].

We can see that, the UV abs curves of vinegar which have same brand are very similar, while the UV abs curves of vinegar which have different brand are of great difference. So we consider processing and analyzing the data by using the method of pattern recognition, and identify the vinegar.

With sampling interval at 0.5nm, a vinegar sample is a vector with 171 dimension, while there are only 7 samples for one kind of vinegar, the dimension of sample vectors is far more than the number of samples, it is probably to appear severe warp when using statistical methods, so we take the interval as 5nm, and the dimension of sample vectors is now 18.

Table 1: The Identification Accuracy of Euclid(Mahalanobis) Distance

| | | | | | |
|----------------------|-----------|-------------|---------|---------|------------|
| Name | Zhenjiang | Xiaoerhei | Shuita | Jiajia | Black Rice |
| of | spicy | grain spicy | mature | mature | spicy |
| vinegar | vinegar | vinegar | vinegar | vinegar | vinegar |
| Euclid distance | 100% | 100% | 100% | 100% | 100% |
| Mahalanobis distance | 100% | 100% | 100% | 100% | 100% |

§3. process and analyze data by pattern recognition method

we randomly choose 5 samples from each kind of vinegar samples for training, the remained 2 samples for testing.

3.1. Euclid(Mahalanobis) distance method

This method is to classify the original samples. Let the i th training sample of the k th kind of vinegar be $X_i^k = (x_{i1}^k, x_{i2}^k, \dots, x_{im}^k)^T$, $k = 1, 2, \dots, 5$, $i = 1, 2, \dots, 5$, $n = 18$, we calculate the mean vector (centre) of each training sample class:

$$m_k = \frac{1}{5} \sum_{i=1}^5 x_i^k \quad (1)$$

and the covariance matrix:

$$s_k = \frac{1}{5} \sum_{i=1}^5 (x_i^k - m_k)(x_i^k - m_k)^T \quad (2)$$

For each testing samples y , we calculate the Euclid distance between y and each centre:

$$d_k^E = [(y - m_k)^T (y - m_k)]^{\frac{1}{2}} \quad (3)$$

and the Mahalanobis distance:

$$d_k^M = [(y - m_k)^T s_k^{-1} (y - m_k)]^{\frac{1}{2}} \quad (4)$$

At last, assign the testing sample y to the 'nearest' class p as the following equation:

$$p = \underset{k}{\operatorname{argmin}} d_k^E \quad \text{or} \quad p = \underset{k}{\operatorname{argmin}} d_k^M \quad (5)$$

We test 10 testing samples by MATLAB7.4, the result is as Table 1:

3.2. Hybrid Discriminant Analysis(HDA) [3]

3.2.1. Method

HDA is to project the original samples data \mathbf{x} (including training and testing samples) into a one-dimension subspace by the following linear transformation:

$$y = \mathbf{w}^T \mathbf{x} \quad (6)$$

Then the classifying is done in the subspace, the testing speed is quicker, and the accuracy of classification is still 100%. HDA is a method based on LDA and PCA. It integrates both discriminant and descriptive information simultaneously, controls the balance between LDA and PCA, it also provides a 2-D parameter space for searching. The objective function of HDA is:

$$\mathbf{w}_{opt} = \underset{\omega}{argmax} \frac{|\mathbf{w}^T[(1-\lambda)S_b + \lambda S_\Sigma]\mathbf{w}|}{|\mathbf{w}^T[(1-\eta)S_\omega + \eta I]\mathbf{w}|} \quad (7)$$

where λ, η are tow parameters ranged from 0 to 1,

$$S_b = \sum_{k=1}^5 5(m_k - \mathbf{m})(m_k - \mathbf{m})^T \quad (8)$$

is the between-class scatter matrix, and $\mathbf{m} = \frac{1}{25}\Sigma \mathbf{x} = \frac{1}{25}\sum_{k=1}^5 5m_k$ is the mean vector of all training samples, while

$$S_\omega = \sum_{k=1}^5 S_k, \text{ and } S_k = \sum_{i=1}^5 (x_i^k - m_k)(x_i^k - m_k)^T \quad (9)$$

is the within-class scatter matrix.

S_Σ is the covariance matrix of all training samples, $S_\Sigma = \frac{1}{25}\Sigma(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T$, \mathbf{I} is unit matrix. According to the Lagrange function method, the solution \mathbf{w} is the largest eigenvector (corresponding the largest eigenvalue) of $[(1-\eta)S_\omega + \eta I]^{-1}[(1-\lambda)S_b + \lambda S_\Sigma]$.

We project all the training and testing samples as (6), let the projection of training sample x_i^k be y_i^k , then the projection centre of each class in subspace is $m_k = \frac{1}{5}\sum_{i=1}^5 y_i^k$, let y be the projection of a testing sample, we calculate the distance between y and the projection centre of each class $d_k = |y - m_k|$, assign the testing sample into the 'nearest' class.

3.2.2. Result and Discussion

Let λ, η be the different parameters between 0 and 1, by using MATLAB7.4, we test 10 testing samples, the result is as Table 2:

We have seen that when $\lambda = 1, \eta = 1$ or $\lambda = 0, \eta = 1$, the result is very satisfying, but it is poor when $\lambda = 0, \eta = 0$.

In fact, when $\lambda = 0, \eta = 0$, (7) becomes:

$$\mathbf{w}_{opt} = \underset{\omega}{argmax} \frac{|\mathbf{w}^T S_b \mathbf{w}|}{|\mathbf{w}^T S_\omega \mathbf{w}|} \quad (10)$$

Table 2: The Identification Accuracy of HAD with Different Parameters

| | | | | | |
|-------------------------|-----------|-------------|---------|---------|------------|
| Name | Zhenjiang | Xiaoerhei | Shuita | Jiajia | Black Rice |
| of | spicy | grain spicy | mature | mature | spicy |
| vinegar | vinegar | vinegar | vinegar | vinegar | vinegar |
| $\lambda = 1, \eta = 1$ | 100% | 100% | 100% | 100% | 100% |
| $\lambda = 0, \eta = 1$ | 100% | 100% | 100% | 100% | 100% |
| $\lambda = 0, \eta = 0$ | 100% | 100% | 50% | 100% | 50% |

This is the objective function of LDA, the solution \mathbf{w} is now the largest eigenvector of $S_\omega^{-1}S_b$. Since the estimate of the scatter matrix S_ω and S_b is based on samples, LDA may not do well in small sample set problem. This conclusion is proved by the result of Table 2.

While $\lambda = 0, \eta = 1$, (7) becomes as following:

$$\mathbf{w}_{opt} = \underset{\omega}{argmax} \frac{|\mathbf{w}^T S_b \mathbf{w}|}{|\mathbf{w}^T \mathbf{I} \mathbf{w}|} \quad (11)$$

The solution \mathbf{w} is now the largest eigenvector of between-class scatter matrix S_b , this shows that (6) makes the scatter degree among each projection class largest, so the distance among each projection class is largest, the result of classification is good of course.

Let $\lambda = 1, \eta = 1$, then (7) becomes as following:

$$\mathbf{w}_{opt} = \underset{\omega}{argmax} \frac{|\mathbf{w}^T S_\Sigma \mathbf{w}|}{|\mathbf{w}^T \mathbf{I} \mathbf{w}|} \quad (12)$$

is the objective function of PCA, \mathbf{w} , which is the solution of the objective function, is the eigenvector associated with the largest eigenvalue of the covariance matrix S_Σ . PAC is statistical analysis method, it can remove the relativity between the elements of the vector, so the components of the transformed vector are disrelated, and be arranged in the order that the corresponding variance are decreased. PAC is superior to LDA in dealing with the small sample set problems because it captures the descriptive information of the data in the projected space. This can be shown in Table 2.

For further analysis, we calculate the nonzero eigenvalues and the accumulated variance cover rate of the covariance matrix.

From the Table 3, we can see that the accumulated variance cover rate of the largest eigenvalue has arrived 93.22%. This shows that the variance contribution rate of the first principal component of the transformed vector has arrived 93.22%, so we obtain the satisfying results by only using the first principal component for classification.

The eigenvector associated with the largest eigenvalue is \mathbf{w} ,
 $\mathbf{w} = (-0.2976 \ -0.3487 \ -0.3781 \ -0.3691 \ -0.3676 \ -0.3157 \ -0.3499 \ -0.3105 \ -0.2069 \ -0.1092 \ -0.0457$

Table 3: The Identification Accuracy of BP Neural Network

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------------|---------|--------|--------|--------|--------|--------|--------|--------|--------|
| Eigenvalues | 20.6706 | 1.4026 | 0.046 | 0.0245 | 0.0141 | 0.0125 | 0.0043 | 0.0001 | 0.0001 |
| Accumulated | | | | | | | | | |
| Cover Rate | 93.22% | 99.54% | 99.75% | 99.86% | 99.92% | 99.98% | 100% | 100% | 100% |

Table 4: The average of the five kinds of vinegar

| wavelet | 245 | 250 | 255 | 260 | 265 | 270 | 275 | 280 | 285 |
|------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| shuita | 0.493 | 0.3582 | 0.3758 | 0.4662 | 0.6036 | 0.7336 | 0.8278 | 0.8432 | 0.7798 |
| jiajia | 3.8294 | 4.3478 | 4.7018 | 4.6418 | 4.7018 | 4.1678 | 4.6024 | 4.119 | 2.8532 |
| zhenjiang | 0.2518 | 0.544 | 0.878 | 1.267 | 1.716 | 2.1162 | 2.3998 | 2.4702 | 2.2708 |
| xiaoerhei | 0.648 | 0.3778 | 0.3382 | 0.3934 | 0.4968 | 0.5884 | 0.6388 | 0.617 | 0.5178 |
| black rice | 0.0488 | 0.1994 | 0.2752 | 0.3256 | 0.3626 | 0.3858 | 0.4016 | 0.3926 | 0.3592 |
| wavelet | 290 | 295 | 300 | 305 | 310 | 315 | 320 | 325 | 330 |
| shuita | 0.669 | 0.5866 | 0.5104 | 0.4202 | 0.32 | 0.2094 | 0.106 | 0.0608 | 0.039 |
| jiajia | 1.6392 | 0.8504 | 0.4188 | 0.2172 | 0.1288 | 0.0758 | 0.0454 | 0.0296 | 0.0228 |
| zhenjiang | 1.8616 | 1.397 | 1.0034 | 0.7098 | 0.4908 | 0.3058 | 0.1506 | 0.0886 | 0.0688 |
| xiaoerhei | 0.3738 | 0.2894 | 0.2264 | 0.1744 | 0.1264 | 0.081 | 0.042 | 0.0264 | 0.0204 |
| black rice | 0.3126 | 0.2638 | 0.2172 | 0.1716 | 0.128 | 0.0852 | 0.044 | 0.0282 | 0.02 |

-0.0125 0.0005 0.0034 0.0030 0.0010 0.0004 0.0001)

We can see that the absolute value of the former 10 elements are bigger than that of the latter 8 elements. The latter 8 elements almost go to zero. These elements correspond to the UV absorbance values which has the wavelength of 295~ 330nm. This shows that it is the absorbency values whose UV wavelength is 245 ~295nm that mainly impact the first principal component , while the impact which generated by the absorbency value with UV wavelength of 295 ~ 330nm can be ignored. From table 4, the average UV curves of 5 kinds of vinegar within the wavelength range of 245 ~ 295nm are of much difference, and within the wave length range of 295 ~ 330nm, the difference is minimal. So in the experiment, we can reduce the range of the scan UV wavelength to 245 ~ 295nm.

3.3. BP Neural Network Method [4]

BP Neural Network method use error back-propagation algorithm. The data of the given sample and with ambiguity relationship can be effective classified. We designed a single hidden layer BP neural network. Since the dimension of the sample vector is 18, so the number of input layer nodes n is 18; As we experiment with 5 kinds of vinegar, the numbers of out layer nodes m is 5 and the hidden layer nodes is $\sqrt{m+n}+a$, where $a \in [1, 10]$ is a constant.

Let the training samples of Black Rice spicy vinegar, Jiajia mature vinegar, Shuita mature vinegar, Xiaoerhei grain spicy vinegar and Zhenjiang spicy vinegar be the input vectors, and

Table 5: The Identification Accuracy of BP Neural Network

| Name of vinegar | Zhenjiang spicy vinegar | Xiaoerhei grain spicy vinegar | Shuita mature vinegar | Jiajia mature vinegar | Black Rice spicy vinegar |
|-------------------------|-------------------------------|-------------------------------------|-----------------------------|-----------------------------|--------------------------------|
| Identification Accuracy | 100% | 100% | 100% | 100% | 100% |

the corresponding output vector be (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1) respectively. In the test, we classify the test sample in the class k if the kth element of the output vector is maximum.

Using sigmoid function $y = \frac{1}{1+e^{-0.5x}}$ as the active function, for the weight and the threshold, we use gradient descent momentum learning method, let the momentum coefficient be $\eta = 0.1$ and the learning rate be $\epsilon = 0.1$.

Input the 25 training samples into the net and let training goal error be 0.01. The training results of MATLAB7.4 shows that when there is 6 hidden notes, the goal error meets our qualifications at 0.00776696 only after 16 times training. Then we input the 10 test samples of given class into the net, compute the number of the samples that are classified correctly, we can obtain the accuracy of the classification, see Table 5.

Conclusion

We studied the small sample sets of 5 kinds of vinegar, by circumrotating evaporation and ultraviolet spectrum scanning, under the conditions of wavelength at 245~330nm, dilution ratio of evaporated liquor at 1:6, evaporation temperature at 45°C, mass concentration of reference at 45g/L. The ultraviolet spectrum curves of vinegar at different storage time are obtained, the data are processed and analyzed by the method of pattern recognition, such as Euclid(Mahalanobis) distance, linear discriminant analysis, principal component analysis, hybrid discriminant analysis and BP neural network. The experiment result shows that the accuracy of recognition was 100% when using Euclid(Mahalanobis) distance, principal component analysis, hybrid discriminant analysis ($\lambda = 0, \eta = 1$) and BP, furthermore, we can reduce the scanning wavelength range of ultraviolet spectrum into 245~295nm. These methods can be effective ways to identify vinegar. The reason of poor recognition accuracy of LDA is that we use the small samples set.

In the future, we will try to identify vinegar by support vector machine [5], [6].

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(σ, τ) -derivations on Jordan ideals

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Abstract Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal and a subring of R . For a (σ, τ) -derivation $d : R \rightarrow R$, we prove the following results: (1) If d is a (σ, σ) -derivation which acts as a homomorphism or as an anti-homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$; (2) If F is a generalized (σ, σ) -derivation which acts as a homomorphism or as an anti-homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$; (3) If d is a (σ, τ) -derivation which acts as a homomorphism on J , then $d = 0$ on R .

Keywords Fuzzy Prime rings, derivations, (σ, τ) -derivations, generalized derivations, generalized (σ, τ) -derivations.

§1. Introduction

Throughout the present paper, R will denote an associative ring with center $Z(R)$. We will write for all $x, y \in R$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie product and Jordan product respectively.

A ring R is said to be 2-torsion-free if whenever $2a = 0$ with $a \in R$, then $a = 0$. A ring R is said to be prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. An additive subgroup J of R is said to be a Jordan ideal of R if $u \circ r \in J$, for all $u \in J$ and $r \in R$. An additive mapping $d : R \rightarrow R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp., $d(x^2) = d(x)x + xd(x)$) holds, for all $x, y \in R$. Let σ, τ are two mappings of R . An additive mapping $d : R \rightarrow R$ is called a (σ, τ) -derivation (resp., Jordan (σ, τ) -derivation) if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ (resp., $d(x^2) = d(x)\sigma(x) + \tau(x)d(x)$) holds, for all $x, y \in R$. Clearly, every $(1, 1)$ -derivation (resp, Jordan $(1, 1)$ -derivation), where 1 is the identity mapping on R is a derivation (resp, Jordan derivation) on R . An additive mapping $F : R \rightarrow R$ is called a generalized derivation associated with a derivation $d : R \rightarrow R$ if $F(xy) = F(x)y + xd(y)$ holds, for all $x, y \in R$. In view of the definition of a (σ, σ) -derivation, the notion of generalized (σ, τ) -derivation can be extended as follows: Let σ, τ are two mappings of R . An additive mapping $F : R \rightarrow R$ is called a generalized (σ, τ) -derivation associated with d if there exists a (σ, τ) -derivation $d : R \rightarrow R$ such that $F(xy) = F(x)\sigma(y) + \tau(x)d(y)$ holds, for all $x, y \in R$. Clearly, every $(1, 1)$ -generalized derivation, where 1 is the identity mapping on R is a generalized derivation. In [3] Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R , then $d = 0$ on R . Further Yenigul and Arac [6] obtained the above result for α -derivation in prime rings. Recently M. Ashraf [2] extended the result for (σ, τ) -derivation in prime and semiprime rings.

As for more details and fundamental results used in this paper without mention we refer to ([1], [4] and [5]).

In the present paper our objective is to extend the above results for a (σ, σ) -derivation which acts as a homomorphism or as anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R , then we will generalize the above extension for a generalized (σ, σ) -derivation.

Finally, we will prove that if $d : R \longrightarrow R$ is a (σ, τ) -derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R , then either $d = 0$ on R or $J \subseteq Z(R)$.

§2. Preliminary Results

We begin with the following lemmas which are essential to prove our main results:

Lemma 2.1. [7] If R is a ring and J a nonzero Jordan ideal of R , then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$.

Lemma 2.2. [7] Let R be a prime ring and J a nonzero Jordan ideal of R . If $a \in R$ and $aJ = 0$ or $Ja = 0$, then $a = 0$.

Lemma 2.3. [7] Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R . If $aJb = 0$, then $a = 0$ or $b = 0$.

Lemma 2.4. [7] Let R be a 2-torsion-free prime ring and J a nonzero Jordan ideal of R . If J is a commutative Jordan ideal, then $J \subseteq Z(R)$.

Lemma 2.5. Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal and a subring of R . Suppose that σ, τ are automorphisms of R . If R admits a (σ, τ) -derivation d such that $d(J) = 0$, then $d = 0$ or $J \subseteq Z(R)$.

Proof. We have $d(u) = 0$, for all $u \in J$. This yields that $d(u \circ r) = 0$, for all $u \in J$ and $r \in R$. Now using the fact that $d(u) = 0$, the above expression yields that

$$\tau(u)d(r) + d(r)\sigma(u) = 0, \text{ for all } u \in J \text{ and } r \in R. \quad (2.1)$$

. Replacing r by rs , $s \in R$ in (2.1) and using (2.1), to get

$$d(r)[\sigma(s), \sigma(u)] - [\tau(r), \tau(u)]d(s) = 0, \text{ for all } u \in J \text{ and } r, s \in R. \quad (2.2)$$

. Replacing s by sv , $v \in J$ in (2.2) and using (2.2), our hypotheses yields that $d(r)\sigma(s)[\sigma(v), \sigma(u)] = 0$, for all $u, v \in J$ and $r, s \in R$ and hence $\sigma^{-1}(d(r))R[v, u] = 0$, for all $u, v \in J$ and $r \in R$. The primeness of R yields that either $d(r) = 0$ or $[v, u] = 0$, for all $u, v \in J$ and $r \in R$. If $[v, u] = 0$, for all $u, v \in J$, then it follows that J is commutative. By using Lemma (2.4), we get $J \subseteq Z(R)$.

§3. (σ, τ) -derivation as a homomorphism or as an anti-homomorphism

Let R be a ring and d a derivation of R . If $d(xy) = d(x)d(y)$ (resp., $d(xy) = d(y)d(x)$) holds, for all $x, y \in R$, then we say that d acts as a homomorphism (resp., anti-homomorphism)

on R .

Bell and Kappe [3] proved that if d is a derivation of a prime ring R which acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal I of R , then $d = 0$ on R .

Further, this result was extended by M. Ashraf for (σ, τ) -derivation in [2] as follows:

Theorem 3.1. [2] Let R be a prime ring, I is a nonzero right ideal of R . Suppose that σ, τ are automorphisms of R and $d : R \longrightarrow R$ is a (σ, τ) -derivation of R .

- (i) If d acts as a homomorphism on I , then $d = 0$ on R .
- (ii) If d acts as an anti homomorphism on I , then $d = 0$ on R .

The purpose of this section is to extend the above study to a (σ, σ) -derivation which acts as a homomorphism or as an anti-homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R , and then we will generalize the above extension for a generalized (σ, σ) -derivation. Finally we will extend the above result for a (σ, τ) -derivation which acts as a homomorphism on a nonzero Jordan ideal and a subring J of a 2-torsion-free prime ring R .

Theorem 3.2. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R . Suppose that σ is an automomorphism of R , and $d : R \longrightarrow R$ is a (σ, σ) -derivation of R .

- (i) If d acts as a homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$.
- (ii) If d acts as an anti-homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$.

proof. Suppose that $J \not\subseteq Z(R)$.

- (i) If d acts as a homomorphism on J , then we have

$$d(uv) = d(u)\sigma(v) + \sigma(u)d(v) = d(u)d(v), \text{ for all } u, v \in J. \quad (3.1)$$

Replacing v by vw , $w \in J$ in (3.1), we get

$$d(u)\sigma(v)\sigma(w) + \sigma(u)(d(v)\sigma(w) + \sigma(v)d(w)) = d(u)(d(v)\sigma(w) + \sigma(v)d(w)).$$

Using (3.1), the above relation yields that $(d(u) - \sigma(u))\sigma(v)d(w) = 0$, for all $u, v, w \in J$, ie, $\sigma^{-1}(d(u) - \sigma(u))v\sigma^{-1}(d(w)) = 0$, for all $u, v, w \in J$, hence, $\sigma^{-1}(d(u) - \sigma(u))J\sigma^{-1}(d(w)) = 0$, for all $u, w \in J$. By using Lemma (2.3), we get either $d(u) - \sigma(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $d(u) - \sigma(u) = 0$, for all $u \in J$, then the relation (3.1) implies that $\sigma(u)d(v) = 0$, for all $u, v \in J$. Now replace u by uw , to get $\sigma(u)\sigma(w)d(v) = 0$, for all $u, v, w \in J$, that is, $uw\sigma^{-1}(d(v)) = 0$, for all $u, v, w \in J$, and hence, $uJ\sigma^{-1}(d(v)) = 0$, for all $u, v \in J$. Thus, by Lemma (2.3), we get either $u = 0$ or $d(v) = 0$, for all $u, v \in J$. But since J is a nonzero Jordan ideal of R , we find that $d(v) = 0$, for all $v \in J$ and hence by Lemma (2.6), we get the required result.

- (ii) If d acts as an anti-homomorphism on J , then we have

$$d(uv) = d(u)\sigma(v) + \sigma(u)d(v) = d(v)d(u), \text{ for all } u, v \in J. \quad (3.2)$$

Replacing u by uv in (3.2), we get

$$(d(u)\sigma(v) + \sigma(u)d(v))\sigma(v) + \sigma(u)\sigma(v)d(v) = d(v)(d(u)\sigma(v) + \sigma(u)d(v)).$$

Using (3.2), the above relation yields that

$$\sigma(u)\sigma(v)d(v) = d(v)\sigma(u)d(v), \text{ for all } u, v \in J. \quad (3.3)$$

Again replace u by $wu, w \in J$ in (3.3), we get

$$\sigma(w)\sigma(u)\sigma(v)d(v) = d(v)\sigma(w)\sigma(u)d(v), \text{ for all } u, v, w \in J. \quad (3.4)$$

In view of (3.3), the relation (3.4) yields that $[d(v), \sigma(w)]\sigma(u)d(v) = 0$, for all $u, v, w \in J$, that is, $\sigma^{-1}([d(v), \sigma(w)])u\sigma^{-1}(d(v)) = 0$, for all $u, v, w \in J$. Hence, $\sigma^{-1}([d(v), \sigma(w)])J\sigma^{-1}(d(v)) = 0$, for all $v, w \in J$.

By using Lemma (2.3), we get either $[d(v), \sigma(w)] = 0$, or $d(v) = 0$, for all $v, w \in J$. If $[d(v), \sigma(w)] = 0$, for all $v, w \in J$, then replacing v by vw in the above relation, we get

$$\sigma(v)[d(w), \sigma(w)] + [\sigma(v), \sigma(w)]d(w) = 0, \text{ for all } v, w \in J. \quad (3.5)$$

Replacing v by $v_1v, v_1 \in J$ in (3.5), and using (3.5), to get $[\sigma(v_1), \sigma(w)]\sigma(v)d(w) = 0$, for all $v, v_1, w \in J$ and hence $[v_1, w]J\sigma^{-1}(d(w)) = 0$, for all $v_1, w \in J$. By Lemma (2.3), we get either $[v_1, w] = 0$ or $d(w) = 0$, for all $v_1, w \in J$.

Now let $J_1 = \{w \in J \mid [v_1, w] = 0, \text{ for all } v_1 \in J\}$ and $J_2 = \{w \in J \mid d(w) = 0\}$. Clearly J_1 and J_2 are additive proper subgroups of J whose union is J .

Hence, by Brauer's trick, either $J = J_1$ or $J = J_2$. If $J = J_1$, then $[v_1, w] = 0$, for all $v_1, w \in J$, that is, J is commutative, and hence by Lemma (2.4), $J \subseteq Z(R)$, a contradiction. On the other hand if $J = J_2$, then by Lemma (2.6), we get the required result.

We generalize the above theorem as follows:

Theorem 3.3. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R . Suppose that σ is an automorphism of R and $F : R \rightarrow R$ is a generalized (σ, σ) -derivation associated with a derivation d .

- (i) If F acts as a homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$.
- (ii) If F acts as an anti-homomorphism on J , then either $d = 0$ on R or $J \subseteq Z(R)$.

Proof. Suppose that $J \not\subseteq Z(R)$.

- (i) If F acts as a homomorphism on J , then we have

$$F(uv) = F(u)\sigma(v) + \sigma(v)d(v) = F(uF(v)), \text{ for all } u, v \in J. \quad (3.6)$$

Replacing v by $vw, w \in J$ in (3.6), we get

$$F(u)\sigma(v)\sigma(w) + \sigma(u)(d(v)\sigma(w) + \sigma(v)d(w)) = F(u)(F(v)\sigma(w) + \sigma(v)d(w)).$$

Using (3.6), the above relation yields that $(F(u) - \sigma(u))\sigma(v)d(w) = 0$, for all $u, v, w \in J$. Hence $\sigma^{-1}(F(u) - \sigma(u))J\sigma^{-1}(d(w)) = 0$, for all $u, w \in J$. Hence by Lemma (2.3), we get either $F(u) - \sigma(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $F(u) - \sigma(u) = 0$, for all $u \in J$, then the relation (3.6) implies that $\sigma(u)d(v) = 0$, for all $u, v \in J$. Now, replace u by uw , to get $\sigma(u)\sigma(w)d(v) = 0$, for all $u, v, w \in J$. This implies that $uw\sigma^{-1}(d(v)) = 0$ and hence $uJ\sigma^{-1}(d(v)) = 0$, for all $u, v \in J$. thus by Lemma (2.3), we get either $u = 0$ or $d(v) = 0$. Since J is a nonzero Jordan ideal of R , we find that $d(v) = 0$, for all $v \in J$ and hence by Lemma (2.6), we get the required result.

- (ii) If F acts as an anti-homomorphism on J , then we have

$$F(uv) = F(u)\sigma(v) + \sigma(u)d(v) = F(v)F(u), \text{ for all } u, v \in J. \quad (3.7)$$

Replacing u by uv in (3.7), we get

$$(F(u)\sigma(v) + \sigma(u)d(v))\sigma(v) + \sigma(u)\sigma(v)d(v) = F(v)(F(u)F(v) + \sigma(u)d(v)).$$

Using (3.7), the above relation yields that

$$\sigma(u)\sigma(v)d(v) = F(v)\sigma(u)d(u), \text{ for all } u, v \in J. \quad (3.8)$$

Again replace u by wu , $w \in J$ in (3.8), to obtain

$$\sigma(w)\sigma(u)\sigma(v)d(v) = F(v)\sigma(w)\sigma(u)d(v), \text{ for all } u, v, w \in J. \quad (3.9)$$

In view of (3.8), the relation (3.9) yields that

$$[F(v), \sigma(w)]\sigma(u)d(v) = 0, \text{ for all } u, v, w \in J.$$

This implies that $\sigma^{-1}([F(v), \sigma(w)])u\sigma^{-1}(d(v)) = 0$, for all $u, v, w \in J$. Hence

$$\sigma^{-1}([F(v), \sigma(w)])J\sigma^{-1}(d(v)) = 0, \text{ for all } v, w \in J.$$

By Lemma (2.3), we get either $[F(v), \sigma(w)] = 0$ or $d(v) = 0$, for all $v, w \in J$.

If $[F(v), \sigma(w)] = 0$, for all $v, w \in J$, then replacing v by vw in the above relation, we get

$$\sigma(v)[d(w), \sigma(w)] + [\sigma(v), \sigma(w)]d(w) = 0, \text{ for all } v, w \in J. \quad (3.10)$$

Now, replacing v by v_1v , v_1 in (3.10) and using (3.10), to obtain $[\sigma(v_1), \sigma(w)]\sigma(v)d(w) = 0$, for all $u, v, v_1 \in J$, hence we get $[v_1, w]J\sigma^{-1}(d(w)) = 0$, for all $v_1, w \in J$. By using Lemma (2.3), we get either $[v_1, w] = 0$ or $d(w) = 0$, for all $v_1, w \in J$. Now let $J_1 = \{w \in J \mid [v_1, w] = 0, \text{ for all } v_1 \in J\}$ and $J_2 = \{w \in J \mid d(w) = 0\}$. Clearly J_1 and J_2 are additive proper subgroups of J whose union is J . Hence by Brauer's trick, either $J = J_1$ or $J = J_2$.

If $J = J_1$, then $[v_1, w] = 0$, for all $v_1, w \in J$, it follows that J is commutative, hence by Lemma (2.4), we get $J \subseteq Z(R)$, a contradiction. On the other hand, if $J = J_2$, then by Lemma (2.6) we get the required result.

In the next theorem we will extend Theorem (3.2) to a (σ, τ) -derivation d of a 2-torsion-free prime ring R which acts as a homomorphism on a Jordan ideal and a subring J of R .

Theorem 3.4. Let R be a 2-torsion-free prime ring, J a nonzero Jordan ideal and a subring of R . Suppose that σ, τ are automorphisms of R and $d : R \rightarrow R$ is a (σ, τ) -derivation of R . If d acts as a homomorphism on J , then $d = 0$ on R .

Proof. Since d acts as a homomorphism on J , then we have

$$d(uv) = d(u)\sigma(v) + \tau(u)d(v) = d(u)d(v), \text{ for all } u, v \in J. \quad (3.11)$$

Replacing v by vw , $w \in J$ in (3.11), we get

$$d(u)\sigma(v)\sigma(w) + \tau(u)(d(v)\sigma(w) + \tau(v)d(w)) = d(u)(d(v)\sigma(w) + \tau(v)d(w)).$$

Using (3.11), the above relation yields that $(d(u) - \tau(u))\tau(v)d(w) = 0$, for all $u, v, w \in J$, this implies that $\tau^{-1}(d(u) - \tau(u))J\tau^{-1}(d(w)) = 0$, for all $u, w \in J$. By using Lemma (2.3), we

get either $d(u) - \tau(u) = 0$ or $d(w) = 0$, for all $u, w \in J$. If $d(u) - \tau(u) = 0$, for all $u \in J$, we obtain $d(u) = \tau(u)$, for all $u \in J$. Then the relation (3.11) implies that $d(u)\sigma(v) + d(u)d(v) = d(u)d(v)$, for all $u, v \in J$, this yields that $d(u)\sigma(v) = 0$, for all $u, v \in J$. Now, replacing v by vw , $w \in J$, we get $d(u)\sigma(v)\sigma(w) = 0$, for all $u, v, w \in J$, that is, $\sigma^{-1}(d(u))vw = 0$, for all $u, v, w \in J$ and hence $\sigma^{-1}(d(u))Jw = 0$, for all $u, w \in J$. Hence by using Lemma (2.3), we get either $d(u) = 0$ or $w = 0$, for all $u, w \in J$. Since J is a nonzero Jordan ideal of R , we have $d(u) = 0$, for all $u \in J$, then by Lemma (2.6), we get $d = 0$ on R .

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Some another remarks on the generalization of Bernoulli and Euler numbers

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Abstract In this paper we investigate generalized Bernoulli and Euler polynomials. In section 1, we enter multiplication theorem for generalized Bernoulli and Euler numbers. Also in section 2, we introduce a matrix representation of B_n^{-1} and E_n^{-1} . Furthermore in section 3 we consider Euler Maclaurin summation formula for α -Bernoulli numbers and in section 4 we give a relation between associated Stirling numbers and $B_n(a, b)$. Also in end section of this paper we consider a new method for representation of Apostol Bernoulli numbers.

Keywords Bernoulli numbers, Euler polynomials, Stirling numbers of the second kind, generating functions.

Short review of classical approaches

Bernoulli numbers were first introduced by Jacques Bernoulli (1654-1705), in the second part of his treatise published in 1713, *Ars conjectandi*, at the time, Bernoulli numbers were used for writing the infinite series expansions of hyperbolic and trigonometric functions. Van den Berg was the first to discuss finding recurrence formulae for the Bernoulli numbers with arbitrary sized gaps (1881) [9]. Ramanujan showed how gaps of size 7 could be found, and explicitly wrote out the recursion for gaps, of size 6 [10]. Lehmer in 1934 extended these methods to Euler numbers, Genocchi numbers, and Lucas numbers (1934) [9], and calculated the 196-th Bernoulli numbers. Bernoulli polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and in classical and numerical analysis. These polynomials can be defined by various methods depending on the applications. In particular, six approaches to the theory of Bernoulli polynomials are known, these are associated with the names of J. Bernoulli, L. Euler, P. E. Appel, A. Hurwitz, E. Lucas and D. H. Lehmer. Also Apostol and Qiu-Ming Luo defined new generalizations of Bernoulli polynomials that we have used in this paper.

§1. Generalized Raabe multiplication theorem

For a real or complex parameter α , the higher order Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the higher order Euler polynomials $E_n^{(\alpha)}(x)$, each of degree n in x as well as in α , are defined

by the following generating functions:

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, (|t| < 2\pi),$$

$$\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, (|t| < 2\pi).$$

That the explicit formula for $B_n^{(\alpha)}(x)$ and $E_n^{(\alpha)}(x)$ are

$$B_n^{(\alpha)} = \sum_{k=1}^n \sigma(n, k) \alpha^k,$$

where

$$\sigma(n, k) = (-1)^{n-k} \frac{n!}{k!} \sum_{\substack{v_1, v_2, \dots, v_k \in N \\ v_1 + v_2 + \dots + v_k = n}} \frac{B_{v_1} B_{v_2} \cdots B_{v_k}}{(v_1 v_2 \cdots v_k) v_1! v_2! \cdots v_k!},$$

and

$$E_n^{(\alpha)}(x) = \sum_{s=0}^n \binom{n}{s} x^{n-s} \sum_{k=0}^s \frac{(-1)^k k!}{2^k} \binom{\alpha + k + 1}{k} S(s, k)$$

respectively. See [1], [2].

Moreover, the higher order Bernoulli numbers $B_n^{(\alpha)}$ and higher order Euler numbers $E_n^{(\alpha)}$ are defined by

$$\left(\frac{t}{e^t - 1}\right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!},$$

and

$$\left(\frac{2}{e^t + 1}\right)^\alpha = \sum_{n=0}^{\infty} E_n^{(\alpha)} \frac{t^n}{n!}$$

respectively.

Clearly, for all nonnegative integers n , the classical Bernoulli and Euler polynomials, $B_n(x)$ and $E_n(x)$ are given by $B_n(x) := B_n^{(1)}(x)$ and $E_n(x) := E_n^{(1)}(x)$.

That the classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined through the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

The explicit formulas for $B_n(x)$ and $E_n(x)$, are

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k},$$

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2-2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k},$$

where $B_k := B_k(0)$ is the k -th Bernoulli number and $E_k := E_k(1)$ is the k -th Euler number.

The Bernoulli numbers may also be calculated from

$$B_n = \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \frac{x}{e^x - 1}.$$

Also the Bernoulli numbers are given by the double sum

$$B_n = \sum_{k=0}^n \frac{1}{k+1} \sum_{r=0}^k (-1)^r \binom{k}{r} r^n.$$

The Bernoulli numbers satisfy the sum

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0.$$

At first we introduce necessary definitions about this matter.

Definition 1.1.([2]) Let $a, b > 0$, $a \neq b$, the generalized Bernoulli numbers $B_n(a, b)$ are defined by

$$\frac{t}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(a, b)}{n!} t^n,$$

where $|t| < \frac{2\pi}{|\ln b - \ln a|}$.

Definition 1.2. Let $a, b > 0$, $a \neq b$, we define the generalized Bernoulli polynomials as

$$\frac{te^{xt}}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(x; a, b)}{n!} t^n,$$

where $|t| < \frac{2\pi}{|\ln b - \ln a|}$.

Definition 1.3. For positive numbers a, b , the generalized Euler numbers $E_k(a, b)$ are defined by

$$\frac{2}{b^{2t} + a^{2t}} = \sum_{n=0}^{\infty} \frac{E_n(a, b)}{n!} t^n.$$

Definition 1.4. For any given positive numbers a, b and $x \in \mathbb{R}$, the generalized Euler polynomials $E_k(x; a, b)$ are defined by

$$\frac{2e^{xt}}{b^t + a^t} = \sum_{n=0}^{\infty} \frac{E_n(x; a, b)}{n!} t^n.$$

Theorem 1.1.([12]) For positive numbers a, b , we have

$$B_n(x+y; a, b) = \frac{1}{2} \sum_{k=0}^n \binom{n}{k} [B_k(y; a, b) + B_k(y+1; a, b)] E_{n-k}(x).$$

Remark 1.1. In special case if we set $b = e, a = 1, y = 0$, then we obtain

$$B_n(x) = \sum_{k=0, k \neq 1}^n \binom{n}{k} B_k E_{n-k}(x),$$

which is the G-S.Cheon formula. (See [1] for detail)

The term Bernoulli polynomials was used first in 1851 by Raabe [10] in connection with the following multiplication theorem

$$\frac{1}{m} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \right) = m^{-n} B_n(mx).$$

Here we give an analogues formula for generalized Bernoulli numbers.

Theorem 1.2. Let $x, y, a, b \in \mathbb{C}$ (Complex numbers) so we have the following identity

$$\frac{1}{m} \sum_{k=0}^{m-1} B_n \left(x + \frac{k}{m} \ln a + \frac{m-k+1}{m} \ln b \right) = m^{-n} B_n(mx).$$

Proof. Let us expand the function

$$\sum_{n=0}^{\infty} \frac{B_n(x, a, b)}{n!} t^n = \frac{t}{b^t - a^t} e^{xt}, b \neq a. \quad (1.1)$$

In powers of x and t and collect the coefficients of $\frac{t^n}{n!}$ as a polynomial $\Psi_n(x, a, b)$ of degree n in x :

$$F(x, t, a, b) = \sum_{n=0}^{\infty} \Psi_n(x, a, b) \frac{t^n}{n!}.$$

Suppose

$$\Psi_n(x, a, b) = A_0^{(n)} x^n + A_1^{(n)} x^{n-1} + \dots + A_n^{(n)}.$$

That

$$A_i^{(n)} := A_i^{(n)}(a, b).$$

If we replace x by $\frac{1}{y}$ and t by ty in (1.1) we get

$$F\left(\frac{1}{y}, ty, a, b\right) = \frac{ty}{b^{ty} - a^{ty}} e^t = \sum_{n=0}^{\infty} y^n \Psi_n\left(\frac{1}{y}, a, b\right) \frac{t^n}{n!}.$$

Letting y tend to zero we obtain

$$\frac{1}{\ln b - \ln a} e^t = \sum_{n=0}^{\infty} A_0^{(n)} \frac{t^n}{n!}.$$

Hence, $A_0^{(n)} = \frac{1}{\ln b - \ln a}$ and hence $\Psi_n(x, a, b)$ is monic.

If in (1.1) we replace x by $x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}$ and sum over k and divide the result by m we get

$$\frac{1}{m} \sum_{k=0}^{m-1} \sum_{n=0}^{\infty} \Psi_n \left(x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}, a, b \right) \frac{t^n}{n!} = \frac{1}{m} \sum_{k=0}^{m-1} F \left(x + \ln a^{\frac{k}{m}} b^{\frac{m-k+1}{m}}, t, a, b \right) = \frac{\frac{t}{m} e^{xt}}{b^{\frac{t}{m}} - a^{\frac{t}{m}}}. \quad (2.1)$$

If, instead, we replace in (1.1) x by mx and t by $\frac{t}{m}$ we obtain

$$F(mx, \frac{t}{m}, a, b) = \frac{\frac{t}{m} e^{xt}}{b^{\frac{t}{m}} - a^{\frac{t}{m}}} = \sum_{n=0}^{\infty} \frac{1}{m^n} \psi_n(mx, a, b) \frac{t^n}{n!}. \quad (3.1)$$

Identifying coefficients of $\frac{t^n}{n!}$ in (2.1), (3.1) we conclude that $\Psi_n(x, a, b)$ satisfies the functional equation

$$\frac{1}{m} \sum_{k=0}^{m-1} \Psi_n \left(x + \frac{k}{m} \ln a + \frac{m-k+1}{m} \ln b \right) = m^{-n} \Psi_n(mx).$$

Because $\Psi_n(x, a, b)$ is monic therefore proof is complete.

Now according to a next lemma we give a representation matrix for B_n^{-1} and E_n^{-1} .

§2. Matrix representation of B_n^{-1} and E_n^{-1}

Lemma 2.1. ([3]) we have

$$\left(\sum_{n=0}^{\infty} a_n x^n \right)^{-1} = \frac{1}{a_0} + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n! a_0^{n+1}} G_n,$$

(let $a_0 \neq 0$),

where

$$G_n = \begin{vmatrix} 2a_1 & a_0 & 0 & 0 & \cdots & 0 \\ 4a_2 & 3a_1 & 0 & 0 & \cdots & 0 \\ 6a_3 & 5a_2 & 3a_0 & 3a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ (2n-2)a_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)a_0 \\ na_n & (n-1)a_{n-1} & \vdots & \vdots & \cdots & a_1 \end{vmatrix}.$$

Now according to previous lemma and because

$$\left(\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \right)^{-1} = \sum_{n=0}^{\infty} \frac{B_n^{(-1)}}{n!} x^n.$$

So if we set $a_n = \frac{B_n}{n!}$ then

$$\sum_{n=0}^{\infty} \frac{B_n^{(-1)}}{n!} x^n = 1 + \sum_{n=1}^{\infty} (-1)^n G_n \frac{x^n}{n!}.$$

Now if $G_n^* := G_n$, $n \geq 1$ and $G_0^* = 1$ so $B_n^{(-1)} = (-1)^n G_n^*$.

So

$$B_n^{(-1)} = (-1)^n \begin{vmatrix} 2B_1 & B_0 & 0 & 0 & \cdots & 0 \\ 2B_2 & 3B_1 & 2B_0 & 0 & \cdots & 0 \\ B_3 & \frac{5}{2}B_2 & 4B_1 & 3B_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{(2n-2)}{(n-1)!}B_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)B_0 \\ \frac{B_n}{(n-1)!} & \frac{B_{n-1}}{(n-2)!} & \vdots & \vdots & \cdots & B_1 \end{vmatrix}.$$

Also

$$\sum_{n=0}^{\infty} \frac{E_n^{(-1)}}{n!} x^n = 1 + \sum_{n=1}^{\infty} (-1)^n G_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n G_n^* \frac{x^n}{n!}.$$

where $G_0^* = 1$

$$E_n^{(-1)} = (-1)^n \begin{vmatrix} 2E_1 & E_0 & 0 & 0 & \cdots & 0 \\ 2E_2 & 3E_1 & 2E_0 & 0 & \cdots & 0 \\ E_3 & \frac{5}{2}E_2 & 4E_1 & 3E_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{(2n-2)}{(n-1)!}E_{n-1} & \vdots & \vdots & \vdots & \cdots & (n-1)E_0 \\ \frac{E_n}{(n-1)!} & \frac{E_{n-1}}{(n-2)!} & \vdots & \vdots & \cdots & E_1 \end{vmatrix}.$$

§3. Euler Maclaurin summation Formula for $B_{n,\alpha}$

Let

$$g_\alpha(z) := 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha},$$

where

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\alpha}}{2^{2k+\alpha} k! \Gamma(\alpha + k + 1)}$$

is the Bessel function of the first kind order α .

The function $\frac{J_\alpha(z)}{z^\alpha}$ is an even entire function of exponential type one, we assume that α is not a negative integer. The zeros $j_k = j_k(\alpha)$ of $\frac{J_\alpha(z)}{z^\alpha}$ may then be ordered in such a way that $j_{-k} = -j_k$ and $0 < |j_1| \leq |j_2| \leq \cdots$. We define a sequence of polynomials $B_{n,\sigma}(x)$ by the generating function

$$\frac{e^{(x-\frac{1}{2})z}}{g_\alpha(iz/2)} = \sum_{n=0}^{\infty} B_{n,\sigma}(x) \frac{z^n}{n!}, |z| < 2|j_1|. \quad (1.3)$$

We call the polynomials $B_{n,\sigma}(x)$ the α -Bernoulli polynomials and $B_{n,\sigma}(0) =: B_{n,\sigma}$ the α -Bernoulli numbers.

To easily we see $B_{0,\sigma}(x) = 1$, $B_{1,\sigma}(x) = x - \frac{1}{2}$, $B_{2,\sigma}(x) = (x - \frac{1}{2})^2 - \frac{1}{8(\alpha+1)}, \dots$. And also to easily of (1.3) we can prove

$$B'_{n,\alpha}(x) = n B_{n-1,\sigma}(x), n = 1, 2, 3, \dots$$

$$B_{n,\sigma}(1-x) = (-1)^n B_{n,\sigma}(x), n = 1, 2, 3, \dots$$

in particular $B_{n,\sigma}(1) = (-1)^n B_{n,\sigma}$. (see [4])

In mathematics, the Euler-Maclaurin formula provides a powerful connection between integrals and sums. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums and infinite series using integrals and the machinery of calculus.

In the context of computing asymptotic expansions of sums and series, usually the most useful form of the Euler-Maclaurin formula is

$$\sum_{k=a}^b g(k) \approx \int_a^b g(x)dx + \frac{g(a) + g(b)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(g^{(2k-1)}(b) - g^{(2k-1)}(a) \right). \text{ (see [5])}$$

Where the symbol indicates that the right-hand side is a so-called asymptotic series for the left-hand side. This means that if we take the first n terms in the sum on the right-hand side, the error in approximating the left-hand side by that sum is at most on the order of the $(n+1)st$ term.

Now we will find a same formula for generalized Bernoulli numbers $B_{n,\sigma}$.

Theorem 3.1.[6] Let f be a real function with continuous $(2k)th$ derivative. Let

$$S_k = \int_1^n f(t)dt + \frac{1}{2} (f(1) + f(n)) + \sum_{i=1}^k \frac{B_{2i}}{(2i)!} \left(f^{(2i-1)}(n) - f^{(2i-1)}(1) \right).$$

Then

$$\sum_{i=1}^n f(i) = S_k - R_k,$$

where the error term is

$$R_k = \int_1^n f^{(2k)}(t) \frac{B_{2k}(\{t\})}{(2k)!} dt$$

with $B_{2k}(t)$ the Bernoulli polynomial and $\{t\} = t - [t]$ the fractional part of t .

Now we consider Euler Maclaurin summation Formula for $B_{n,\sigma}$,

The technique employs repeated integration by formula

$$B'_{n,\alpha}(x) = nB_{n-1,\alpha}, n = 1, 2, 3, \dots$$

to create new derivatives. We start with

$$\int_0^1 f(x)dx = \int_0^1 f(x)B_{0,\alpha}(x)dx. \quad (2.3)$$

Because $B'_{1,\alpha}(x) = B_{0,\alpha}(x) = 1$ substituting $B'_{1,\alpha}(x)$ in (2.3) and integrating by parts, we obtain

$$\begin{aligned} \int_0^1 f(x)dx &= f(1)B_{1,\alpha}(1) - f(0)B_{1,\alpha}(0) - \int_0^1 f'(x)B_{1,\alpha}(x)dx \\ &= \frac{f(1) + f(0)}{2} - \int_0^1 f'(x)B_{1,\alpha}(x)dx. \end{aligned}$$

Again we have $B_{1,\alpha}(x) = \frac{1}{2}B'_{2,\alpha}(x)$ and integrating by parts

$$\int_0^1 f(x)dx = \frac{f(1)+f(0)}{2} - \frac{1}{2!} \left[f'(1)B_{2,\alpha}(1) - f'(0)B_{2,\alpha}(0) \right] + \frac{1}{2!} \int_0^1 f^{(2)}(x)B_{2,\alpha}(x)dx.$$

Using the relation

$$B_{n,\alpha}(1) = (-1)^n B_{n,\alpha}(0) = (-1)^n B_{n,\alpha}, (n = 0, 1, 2, 3, \dots)$$

And continuing this process , we have

$$\int_0^1 f(x)dx = \frac{f(1)+f(0)}{2} - \sum_{p=1}^q \frac{1}{p!} B_{p,\alpha} \left[(-1)^p f^{(p-1)}(1) - f^{(p-1)}(0) \right] + \frac{1}{q!} \int_0^1 f^{(q)}(x)B_{q,\alpha}(x)dx, \quad (3.3)$$

This is the generalization of Euler-maclaurin integration formula , it assume that the function $f(x)$ has the required derivatives. The rang of integration in (2.2) my be shifted $[0,1]$ to $[1,2]$ by replacing $f(x)$ by $f(x+1)$. Adding such results up to $[n-1, n]$,

$$\begin{aligned} \int_0^n f(x)dx &= \frac{1}{2}f(0) + f(1) + f(2) + \dots + f(n-1) + \frac{1}{2}f(n) \\ &- \sum_{p=1}^q \frac{1}{p!} B_{p,\alpha} \left[(-1)^p f^{(p-1)}(n) - f^{(p-1)}(0) \right] + \frac{1}{q!} \int_0^1 B_{q,\alpha}(x) f^{(q)}(x+v)dx. \end{aligned}$$

The terms

$$\frac{1}{2}f(0) + f(1) + f(2) + \dots + f(n-1) + \frac{1}{2}f(n)$$

appear exactly as in trapezoidal integration or quadrature .

§4. Identity for Apostol Bernoulli numbers

Definition 4.1. The Apostol Bernoulli numbers $\beta_n(\lambda)$ are defined by means of the generating functions

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!}, |t + \log \lambda| < 2\pi.$$

That $\beta_n(\lambda)$ is called Apostol Bernoulli numbers.

Lemma 4.1.([6]) Suppose that $|x| < 1$ so we have

$$\sum_{k=0}^{\infty} f(k)x^k = - \sum_{m=0}^{\infty} \frac{f^{(m-1)}(0)}{m!} \beta_m(x), |x| < 1.$$

Now according to pervious lemma we give one identity for Apostol Bernoulli numbers.

Corollary 4.1. Suppose that $|x| < 1$ so we have

$$1) \sum_{k=0}^{\infty} \cos kx^k = \sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n-1}(x)}{(2n-1)!} = \frac{1 - x \cos 1}{1 - 2x \cos 1 + x^2}, x \neq 0, |x| < 1;$$

$$2) \sum_{k=0}^{\infty} \sin kx^k = \sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n}(x)}{(2n)!} = \frac{x \sin 1}{1 - 2x \cos 1 + x^2}, x \neq 0, |x| < 1.$$

Proof. If in lemma (2.1) we set $f(x) = e^{ix}$ [where $i^2 = -1$] we get

$$\sum_{k=0}^{\infty} e^{ik} x^k = - \sum_{m=1}^{\infty} \frac{i^{m-1}}{m!} \beta_m(x), |x| < 1.$$

So because $e^{ik} = \cos k + i \sin k$ we get

$$\sum_{k=0}^{\infty} \cos kx^k + i \sum_{k=0}^{\infty} \sin kx^k = \sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n-1}(x)}{(2n-1)!} + i \sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n}(x)}{(2n)!}, |x| < 1.$$

Also we have

$$\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r \cos \theta}{1 - 2r \cos \theta + r^2},$$

and

$$\sum_{n=0}^{\infty} r^n \sin n\theta = \frac{1 - r \sin \theta}{1 - 2r \cos \theta + r^2}.$$

So

$$\sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n-1}(x)}{(2n-1)!} = \frac{1 - x \cos 1}{1 - 2x \cos 1 + x^2},$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n \beta_{2n}(x)}{(2n)!} = \frac{x \sin 1}{1 - 2x \cos 1 + x^2}.$$

Therefore proof is complete.

§5. Asymptotic relation between 2-associated stirling number and $B_n(a, b)$

We defined the generalized 2-associated stirling numbers by

$$\sum_{n=k}^{\infty} S_2^*(n, a, b, k) \frac{t^n}{n!} = \frac{b^t - (1+t)a^t}{a^{kt} k!}, a \neq 0,$$

where k and r are positive integers. It is clear that if we set $b = e$ and $a = 1$ then

$$\sum_{n=k}^{\infty} S_2^*(n, k) \frac{t^n}{n!} = \frac{(e^t - 1 - t)^k}{k!}, \text{ see [7].}$$

We give asymptotic expansion of certain sums for generalized 2-associated stirling numbers of the second kind, Bernoulli numbers, Euler numbers by Darboux's method.

Lemma 5.1. Assume that

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

is an analytic function for $|t| < r$ and with a finite number of algebraic singularities on the circle $|t| = r$, $\alpha_1, \alpha_2, \dots, \alpha_l$ are singularities of order ω is the highest order of all singularities. Then

$$a_n = \left(\frac{n^{\omega-1}}{\Gamma(\omega)} \right) \left(\sum_{k=1}^l g_k(\alpha_k) \alpha_k^{-n} + O(r^{-n}) \right), \text{ see [8].}$$

Where $\Gamma(w)$ is the gamma function, and

$$g_k(\alpha_k) = \lim_{t \rightarrow \alpha_k} \left(1 - \left(\frac{t}{\alpha_k} \right) \right)^\omega f(t).$$

Theorem 5.1. Suppose that $n \geq 1$ and $k \geq 1$, where k is fixed, when $n \rightarrow \infty$, we have (here let $\ln \frac{b}{a}$ is a algebraic number)

$$\sum_{p+q=2n} \frac{S_2^*(p+k, a, b, k) B_q(a, b)}{(p+k)! q!} \approx \frac{2(-1)^{n+k+1}}{\left(\frac{2\pi}{|\ln b - \ln a|} \right)^{2n} k!}.$$

Proof. It is clear that according to definition we have

$$\sum_{p=0}^{\infty} S_2^*(p+k, a, b, k) \frac{t^p}{(p+k)!} \sum_{q=0}^{\infty} B_q(a, b) \frac{t^q}{q!} = \frac{(b^t - (1+t)a^t)^k}{k! t^{k-1} (b^t - a^t) a^{kt}}, a \neq b.$$

Let

$$f(t) = \frac{(b^t - (1+t)a^t)^k}{k! t^{k-1} (b^t - a^t) a^{kt}},$$

then $f(t)$ is analytic for $|t| < \frac{2\pi}{|\ln b - \ln a|}$ and with two algebraic singularities on the circle $|t| = \frac{2\pi}{|\ln b - \ln a|}$.

$\alpha_1 = \frac{2\pi i}{|\ln b - \ln a|}$ and $\alpha_2 = \frac{-2\pi i}{|\ln b - \ln a|}$ are singularities of order 1. To easily we can compute

$$\lim_{t \rightarrow \frac{2\pi i}{|\ln b - \ln a|}} \left(1 - \frac{t}{\frac{2\pi i}{|\ln b - \ln a|}} \right) f(t) = \lim_{t \rightarrow \frac{-2\pi i}{|\ln b - \ln a|}} \left(1 + \frac{t}{2\pi i} \right) f(t) = \frac{(-1)^{k+1}}{k! |\ln b - \ln a|}.$$

It follows from [8] that

$$\begin{aligned} \sum_{p+q=2n} \frac{S_2^*(p+k, a, b, k) B_q(a, b)}{(p+k)! q!} &= \frac{1}{\Gamma(1)} \left\{ \frac{(-1)^{k+1}}{k!} \left[\left(\frac{2\pi!}{|\ln b - \ln a|} \right)^{-n} \right. \right. \\ &\quad \left. \left. + \left(\frac{-2\pi!}{|\ln b - \ln a|} \right)^{-n} \right] + O \left(\left(\frac{-2\pi!}{|\ln b - \ln a|} \right)^{-n} \right) \right\}. \end{aligned}$$

So we have

$$\sum_{p+q=2n} \frac{S_2^*(p+k, a, b, k) B_q(a, b)}{(p+k)! q!} \approx \frac{(-1)^{k+1} [(i)^{2n} + (-i)^{2n}]}{\left(\frac{2\pi}{|\ln b - \ln a|} \right)^{2n} k!} = \frac{2(-1)^{n+k+1}}{\left(\frac{2\pi}{|\ln b - \ln a|} \right)^{2n} k!}.$$

Therefore proof is complete.

§6. New method for representation of Apostol Bernoulli and Euler polynomials

Let $\sigma(n)$ denote the set of partitions of n (a nonnegative integer) usually denoted by $1^{k_1} 2^{k_2} 3^{k_3} \dots n^{k_n}$ with $\sum i k_i = n$

For nonnegative integral vector $\bar{k} = (k_1, k_2, \dots, k_n)$, the multinomial coefficient $\left(\frac{x}{\bar{k}}\right)$ as usual, is defined by

$$\left(\frac{x}{\bar{k}}\right) = \frac{(x)_{|\bar{k}|}}{\prod (k_i)!},$$

where the finite product \prod runs over i from 1 to n , $(x)_k$ stands for the all factorial notation, and $|\bar{k}|$ represents the coordinate sum for the vector $\bar{k} = (k_1, k_2, \dots, k_n)$. Now let

$$[g(t)]^x = \sum_{n \geq 0} A_n(x) t^n,$$

where x is an arbitrary complex number independent of t .

Theorem 6.1. ([11]) For arbitrary complex number x and y ,

$$A_n(xy) = \sum_{\sigma(n)} \left(\frac{x}{\bar{k}}\right) \prod [A_i(y)]^{k_i}$$

So according to these theorems we have the following results.

Corollary 6.1. Let α, β are complex numbers then we have the following identities

$$B_n^{(\alpha\beta)} = n! \sum_{\sigma(n)} \left(\frac{\alpha}{\bar{k}}\right) \prod \left(\frac{B_i^{(\beta)}}{i!}\right)^{k_i},$$

$$E_n^{(\alpha\beta)} = n! \sum_{\sigma(n)} \left(\frac{\alpha}{\bar{k}}\right) \prod \left(\frac{E_i^{(\beta)}}{i!}\right)^{k_i}.$$

Proof. Let $A_n(\alpha) = \frac{B_n^{(\alpha)}}{n!}$ in theorem 6.1.

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The background is a deep red with a subtle texture. On the left, there are intricate, light red swirling lines. On the right, there is a large, stylized, light red figure that appears to be a person or a decorative element. A bright, vertical light beam or lens flare effect runs down the right side of the image.

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